Modal Testing

(Lecture 1)

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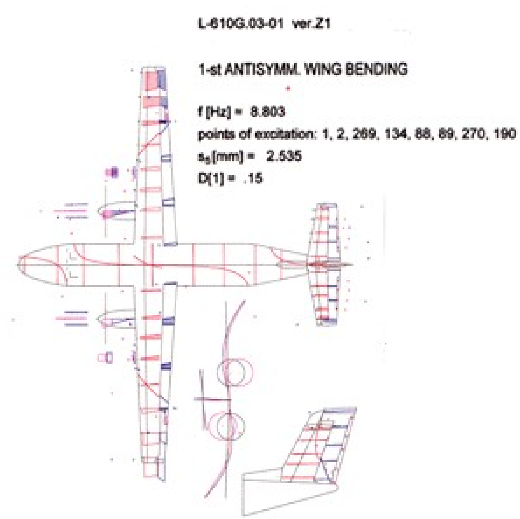
Overview

- Introduction to Modal Testing
- Applications of Modal Testing
- Philosophy of Modal Testing
- Summary of Theory
- Summary of Measurement Methods
- Summary of Modal Analysis Processes
- Review of Test Procedures and Levels

Introduction to Modal Testing

- Experimental Structural Dynamics
  - To understand and to control the many vibration phenomenon in practice
    - Structural integrity (Turbine blades- Suspension Bridges)
    - Performance (malfunction, disturbance, discomfort)

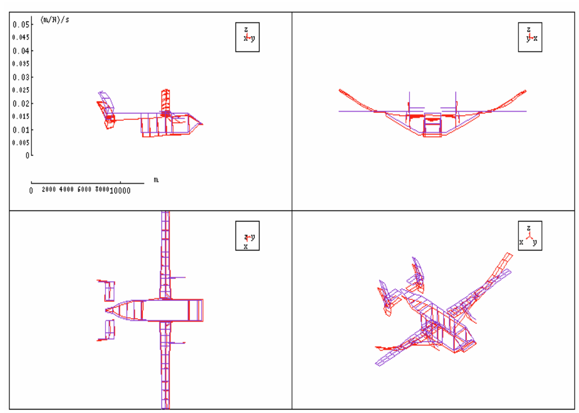


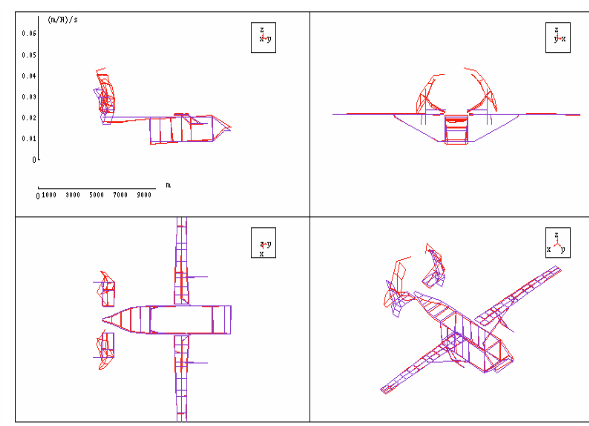
Introduction to Modal Testing (continued)

- Necessities for experimental observations
  - Nature and extend of vibration in operation
  - Verifying theoretical models
  - Material properties under dynamic loading (damping capacity, friction,…)

Introduction to Modal Testing (continued)

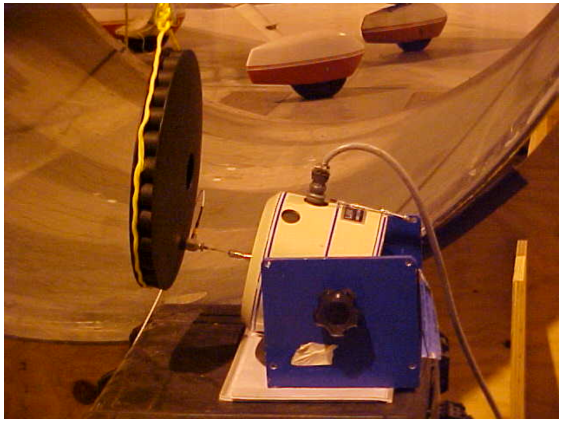
- Test types corresponding to objectives:
  - Operational Force/Response measurements
    - Response measurement of PZL Mielec Skytruck Mode Shapes (3.17 Hz, 1.62 %), (8.39 Hz, 1.93 %)





Introduction to Modal Testing (continued)

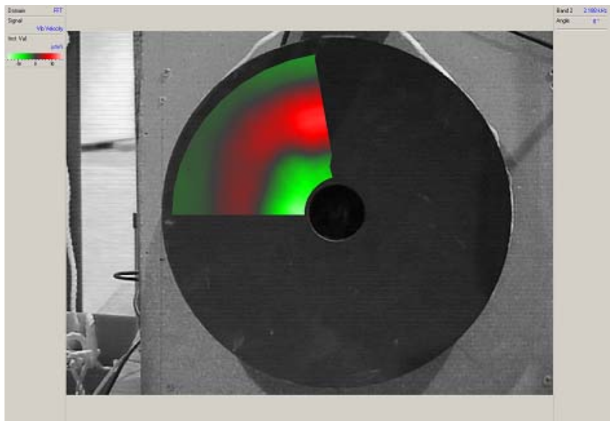
- Modal Testing in a controlled environment/ Resonance Testing/ Mechanical Impedance Method
  - Testing a component or a structure with the objective of obtaining mathematical model of dynamical/vibration behavior
  - Structural Analysis of ULTRA Mirror

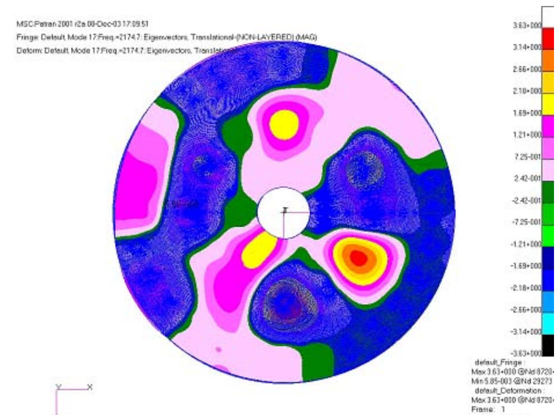


Introduction to Modal Testing (continued)

- Milestones in the development:
  - Kennedy and Pancu (1947)
    - Natural frequencies and damping of aircrafts
  - Bishop and Gladwell (1962)
    - Theory of resonance testing
  - ISMA (bi-annual since 1975)
  - IMAC (annual since 1982)

Applications of Modal Testing





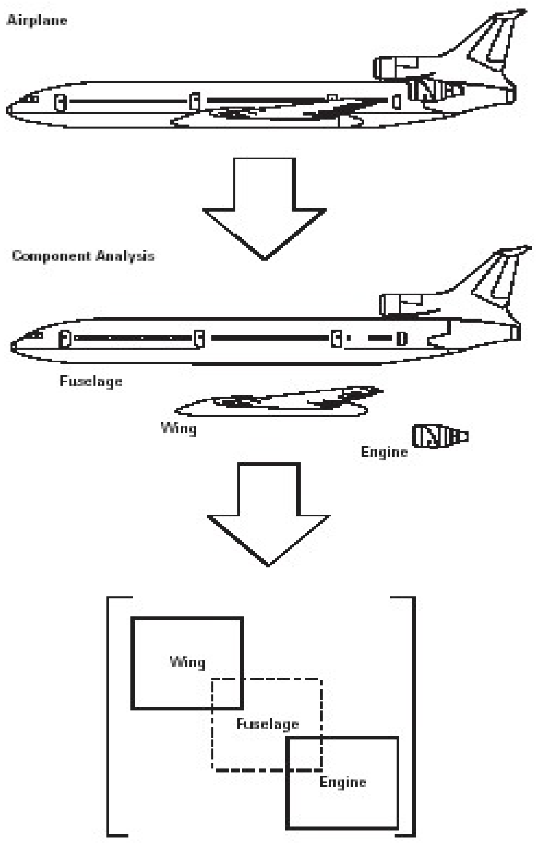
- Model Validation/Correlation:
  - Producing major test modes validates the model
    - Natural frequencies
    - Mode shapes
    - Damping information are not available in FE models

Applications of Modal Testing (continued)

- Model Updating
  - Correlation of experimental/analytical model
  - Adjust/correct the analytical model
  - Optimization procedures are used for updating.

Applications of Modal Testing (continued)

- Component Model Identification
  - Substructure process
  - The component model is incorporated into the structural assembly



Applications of Modal Testing (continued)

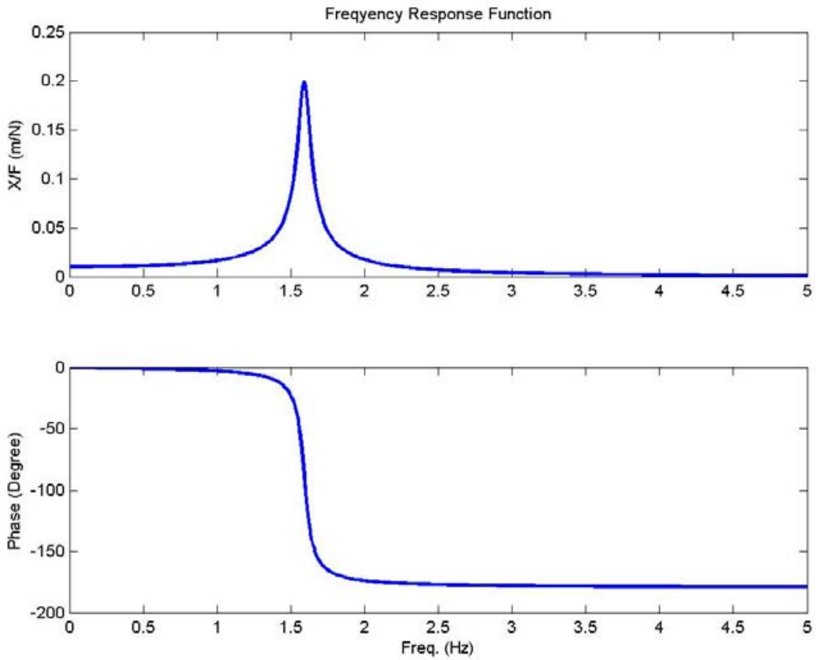
- Force Determination
  - Knowledge of dynamic force is required
  - Direct force measurement is not possible
  - Measurement of response + Analytical Model results the external force

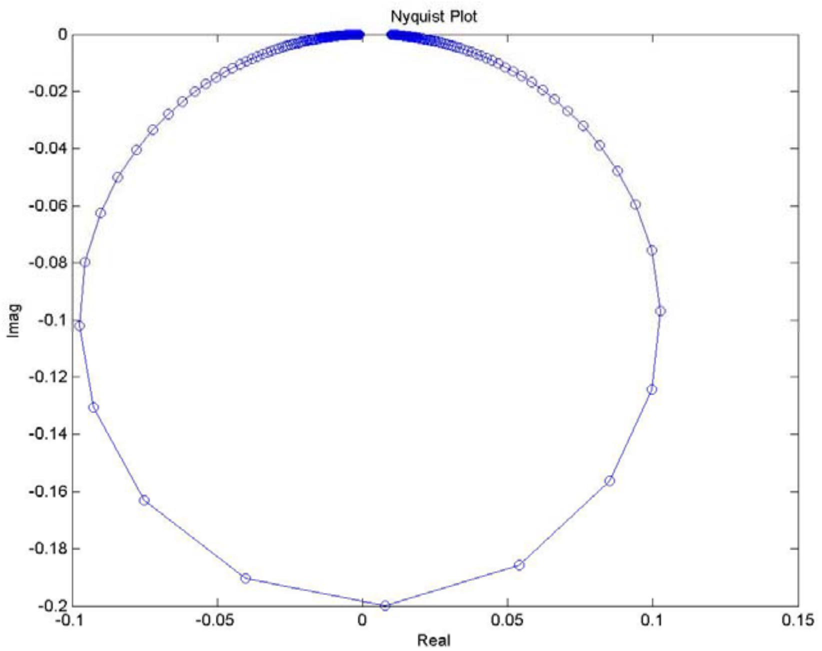
$$\left([K]-\omega^2[M]\right)\{x\}=\{f\}$$

Philosophy of Modal Testing

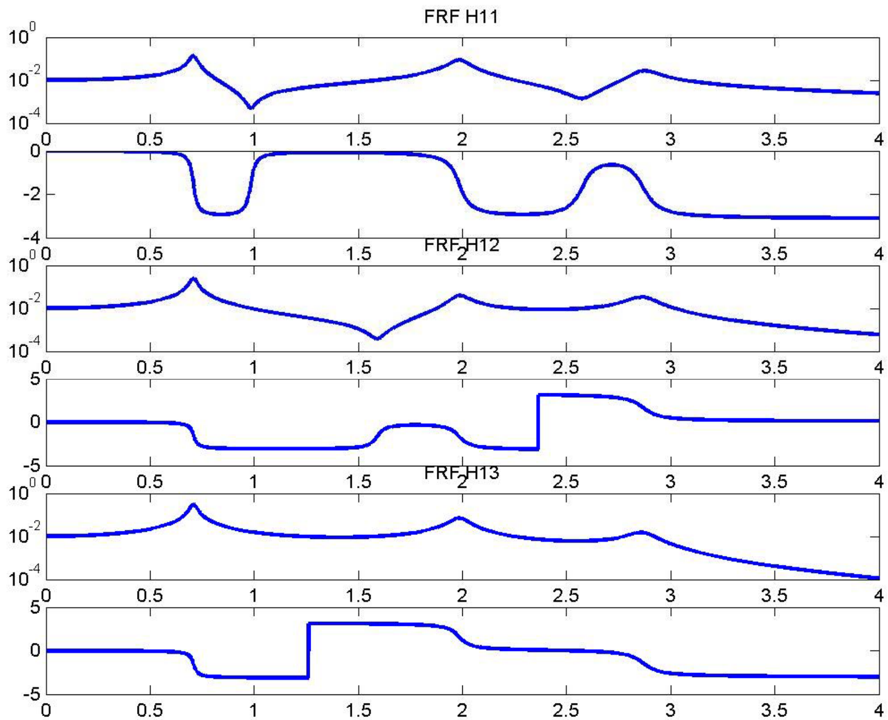
- Integration of three components:
  - Theory of vibration
  - Accurate vibration measurement
  - Realistic and detailed data analysis
- Examples:
  - Quality and suitability of data for process
  - Excitation type
  - Understanding of forms and trends of plots
  - Choice of curve fitting
  - Averaging

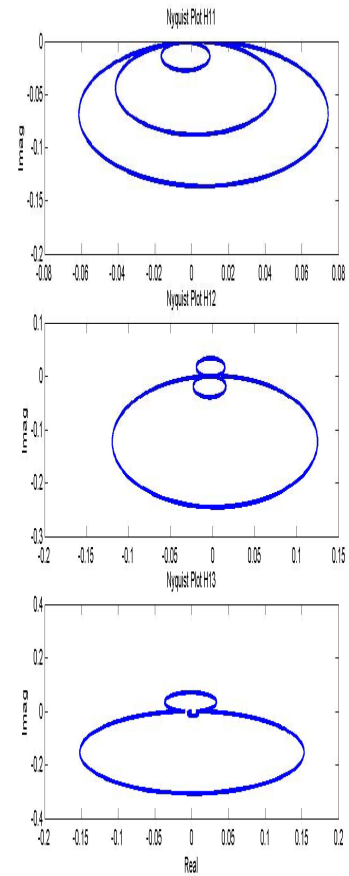
Summary of Theory (SDOF)





Summary of Theory (MDOF)





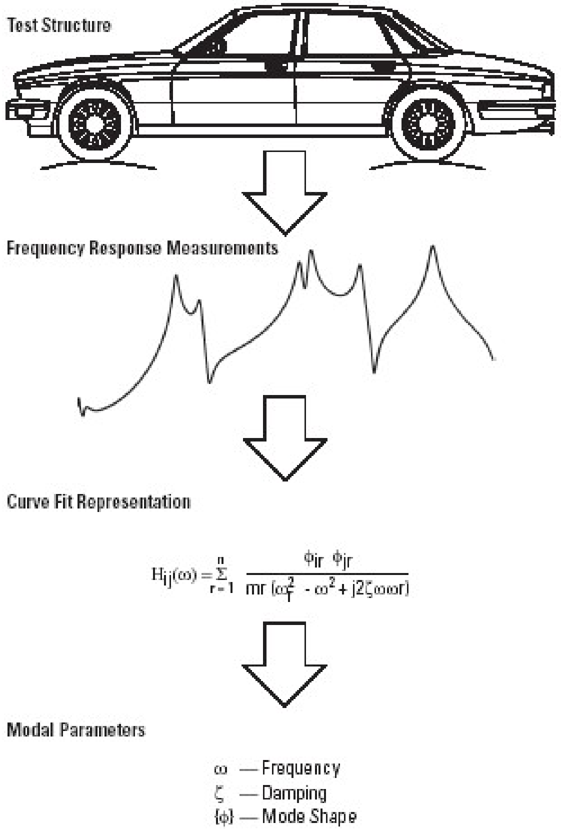
Summary of Theory

- Definition of FRF:

$$H(\omega) = \left([K] - \omega^2 [M] + i[D]\right)^{-1}$$

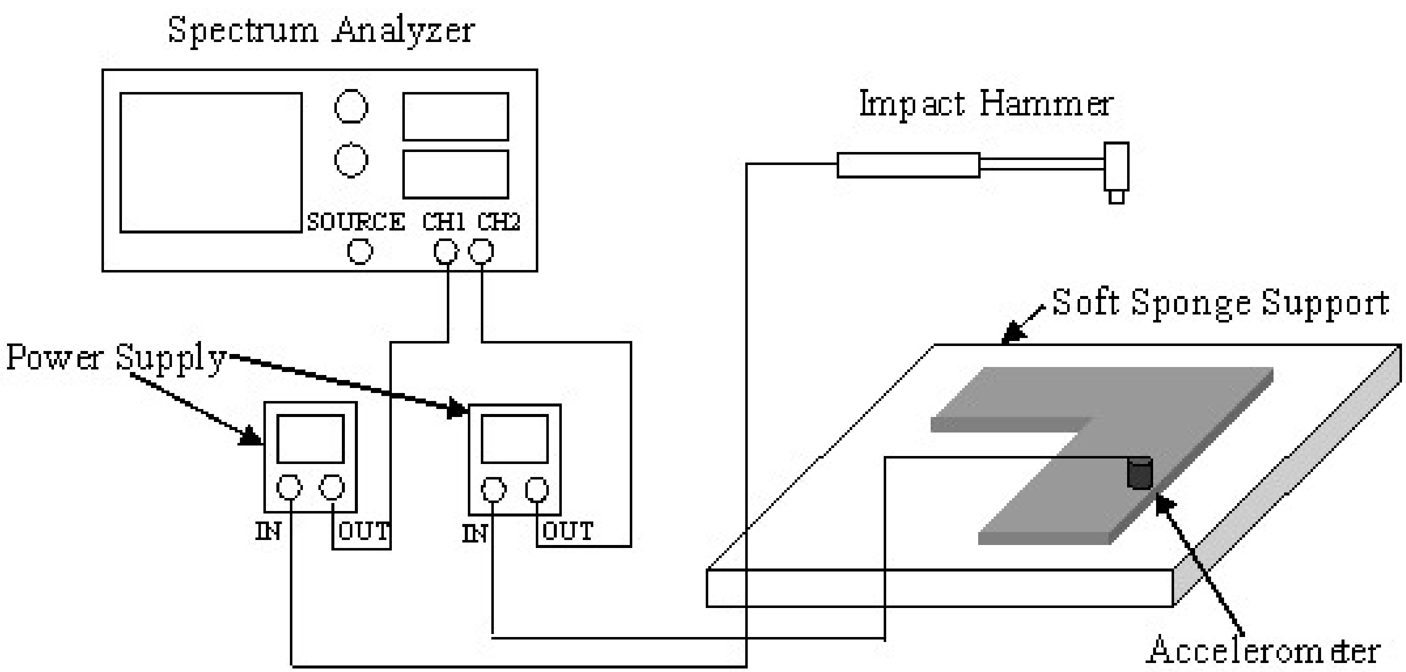
$$h_{jk}(\omega) = \frac{x_j(\omega)}{f_k(\omega)} = \sum_{r=1}^{N} \frac{\phi_{jr}\phi_{kr}}{\omega_r^2 - \omega^2}.$$

- Curve-fitting the measured FRF:
  - Modal Model is obtained
  - Spatial Model is obtained



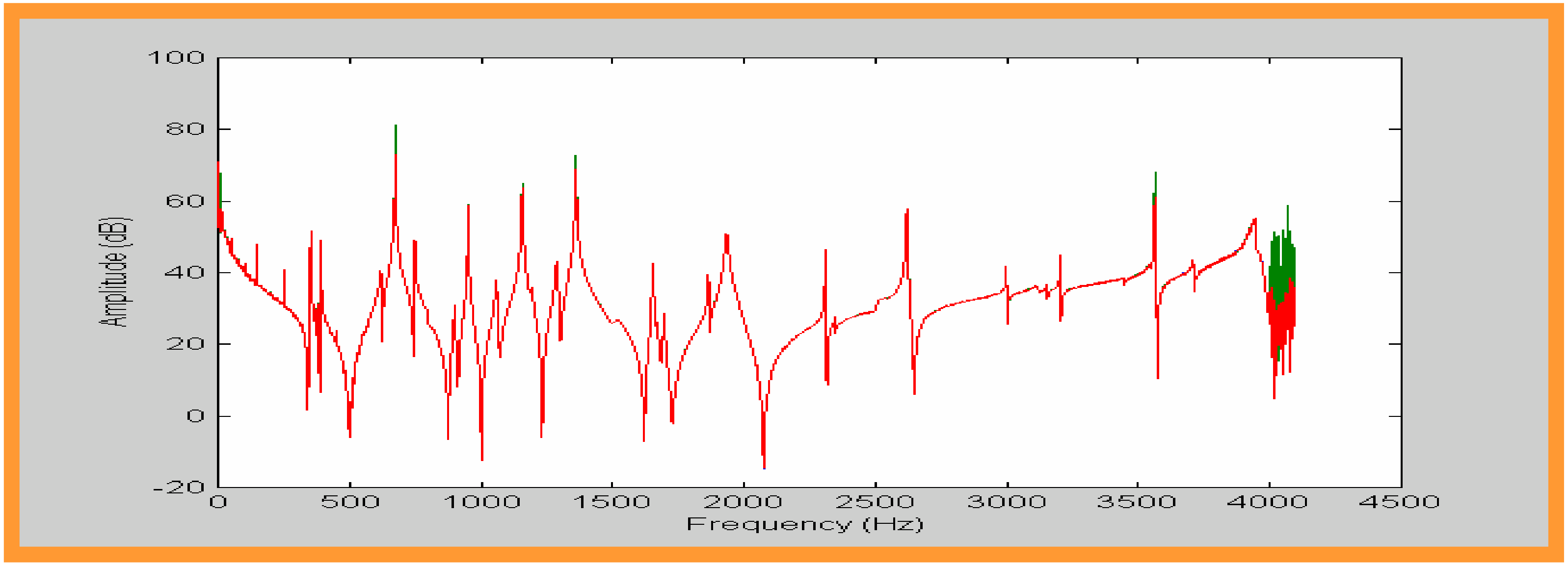
Summary of Measurement Methods

- Basic measurement system:
  - Single point excitation



Summary of Modal Analysis Processes

- Analysis of measured FRF data
  - Appropriate type of model (SDOF,MDOF,…)
  - Appropriate parameters for chosen model



Review of Test Procedures and Levels

- The procedure consists of:
  - FRF measurement
  - Curve-Fitting
  - Construct the required model
- Different level of details and accuracy in above procedure is required depending on the application.

Review of Test Procedures and Levels

- Levels according to Dynamic Testing Agency:

Level	Natural Freq	Damping ratio	Mode Shapes	Usable for validation	Out of range residues	Updating
0						
1			Only in few points			
2						
3						
4						

Text Books

- Ewins, D.J. , 2000, “Modal Testing; theory, practice and application”, 2nd edition, Research studies press Ltd.
- McConnell, K.G., 1995, “Vibration testing; theory and practice”, John Wiley & Sons.
- Maia, *et. al.* , 1997, “Theoretical and Experimental Modal Analysis”, Research studies press Ltd.

Evaluation Scheme

- Home Works (20%)
- Mid-term Exam (20%)
- Course Project (30%)
- Final Exam (30%)

Modal Testing

(Lecture 10)

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Theoretical Basis

- Analysis of weakly nonlinear structures
- Approximate analysis of nonlinear structures
- Cubic stiffness nonlinearity
- Coulomb friction nonlinearity
- Other nonlinearities and other descriptions

Analysis of weakly nonlinear structures

- The whole bases of **modal testing** assumes linearity:
  - Response linearly related to the excitation
  - Response to simultaneous application of several forces can be obtained by superposition of responses to individual forces
- An introduction to characteristics of weakly nonlinear systems is given to detect if any nonlinearity is involved during modal test.

Cubic stiffness nonlinearity

$$m\ddot{x}+c\dot{x}+kx+k_3x^3=F\sin(\omega t-\phi)$$

$$\Rightarrow x(t)=X\sin(\omega t)$$

$$\Rightarrow -m\omega^2X\sin(\omega t)+c\omega X\cos(\omega t)+kX\sin(\omega t)+k_3X^3\sin^3(\omega t)$$
$$=F\sin(\omega t-\phi)$$

$$\Rightarrow -m\omega^2X\sin(\omega t)+c\omega X\cos(\omega t)+kX\sin(\omega t)+$$
$$k_3X^3\left(\frac{3}{4}\sin(\omega t)-\frac{1}{4}\sin(3\omega t)\right)=F\sin(\omega t-\phi)$$

Cubic stiffness nonlinearity

$$-m\omega^2 X\sin(\omega t)+c\omega X\cos(\omega t)+kX\sin(\omega t)+$$

$$k_3X^3\left(\frac{3}{4}\sin(\omega t)-\frac{1}{4}\sin(3\omega t)\right)=$$

$$F\sin(\omega t)\cos(\phi)-F\cos(\omega t)\sin(\phi)$$

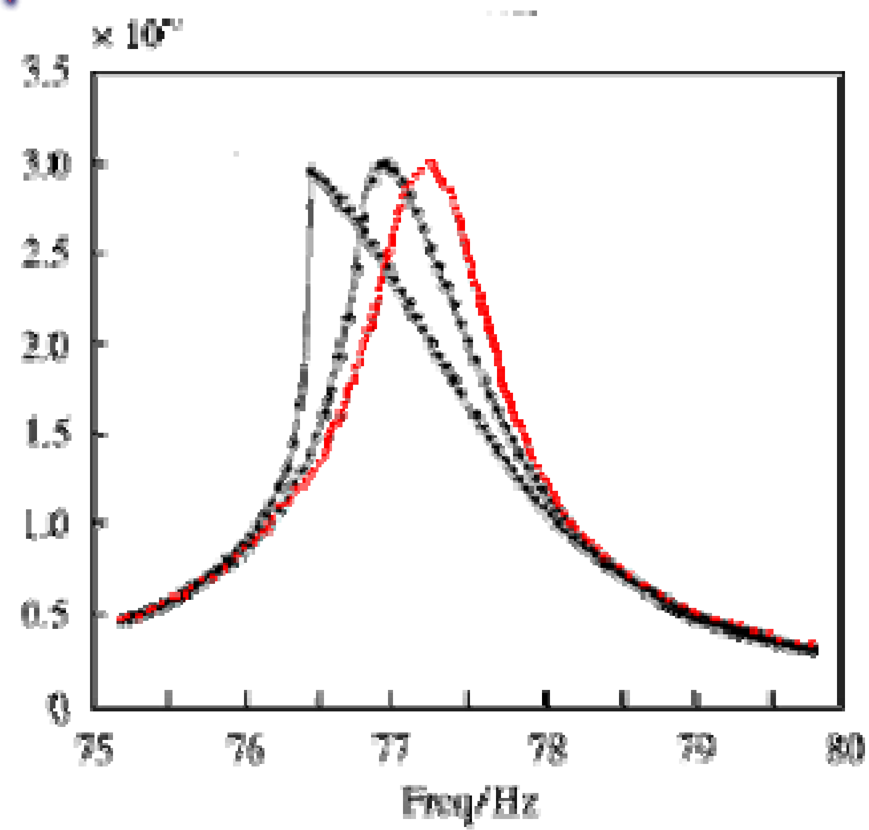
$$\Rightarrow\begin{cases}-m\omega^2 X+kX+\dfrac{3}{4}k_3X^3=F\cos(\phi)\\ c\omega X=-F\sin(\phi)\end{cases}$$

Cubic stiffness nonlinearity

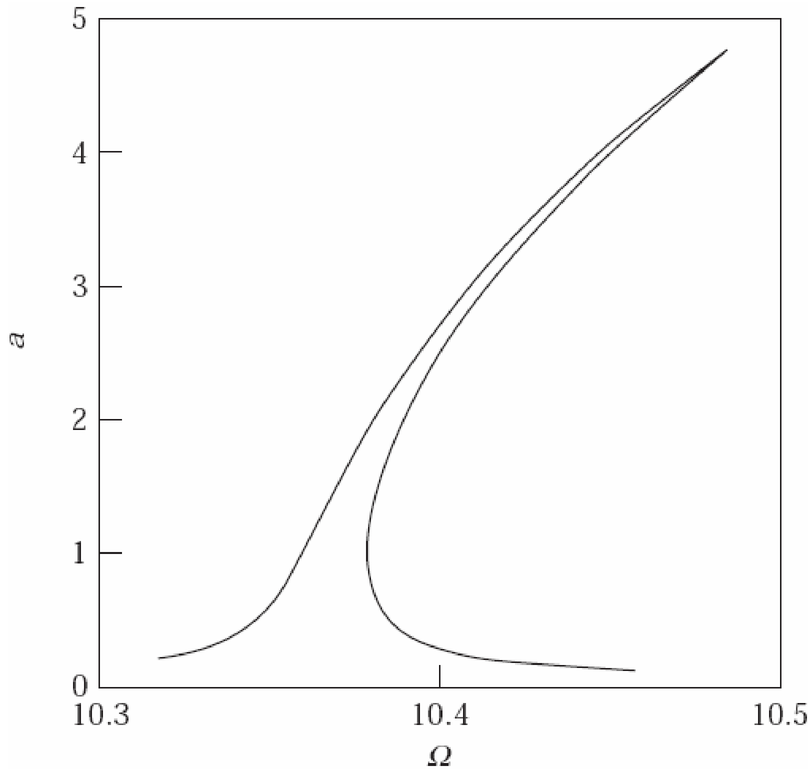
$$\left|\frac{X}{F}\right| = \frac{1}{\sqrt{\left(-m\omega^2 + k + \frac{3}{4}k_3 X^2\right)^2 + (c\omega)^2}}$$

$$k_{eq} = k + \frac{3}{4}k_3 X^2$$

Cubic stiffness nonlinearity

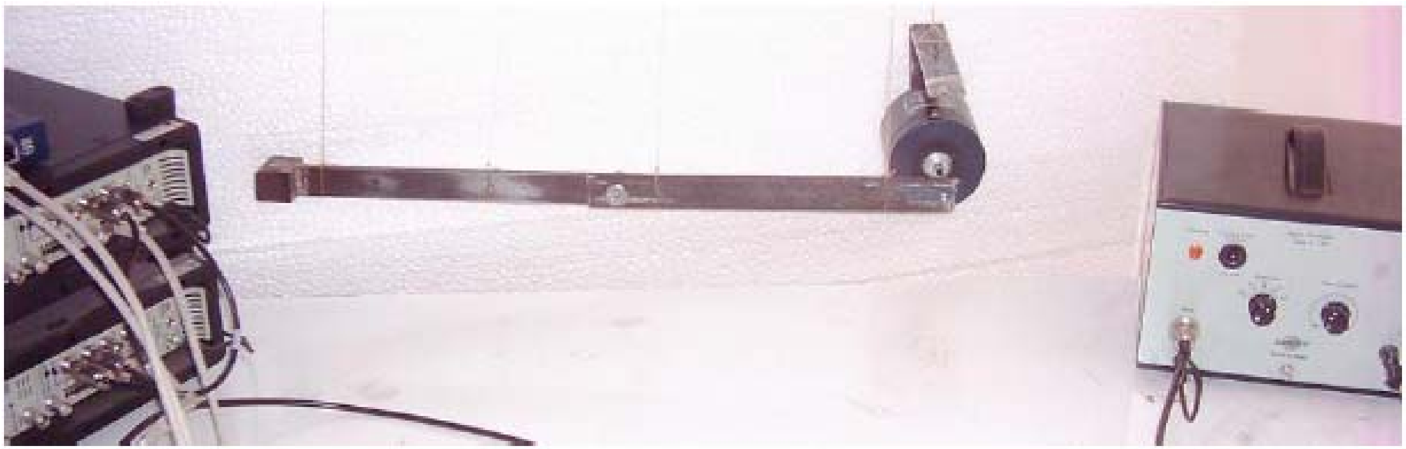


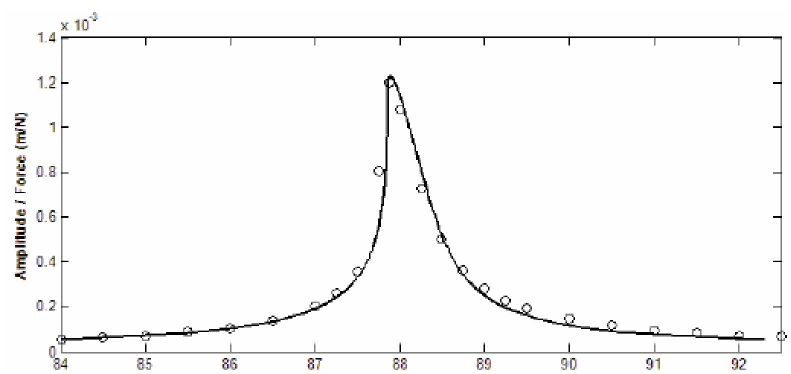
Softening effect

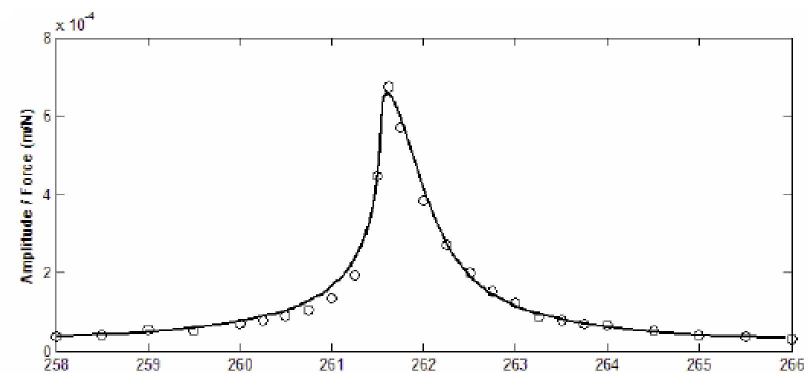


Hardening effect

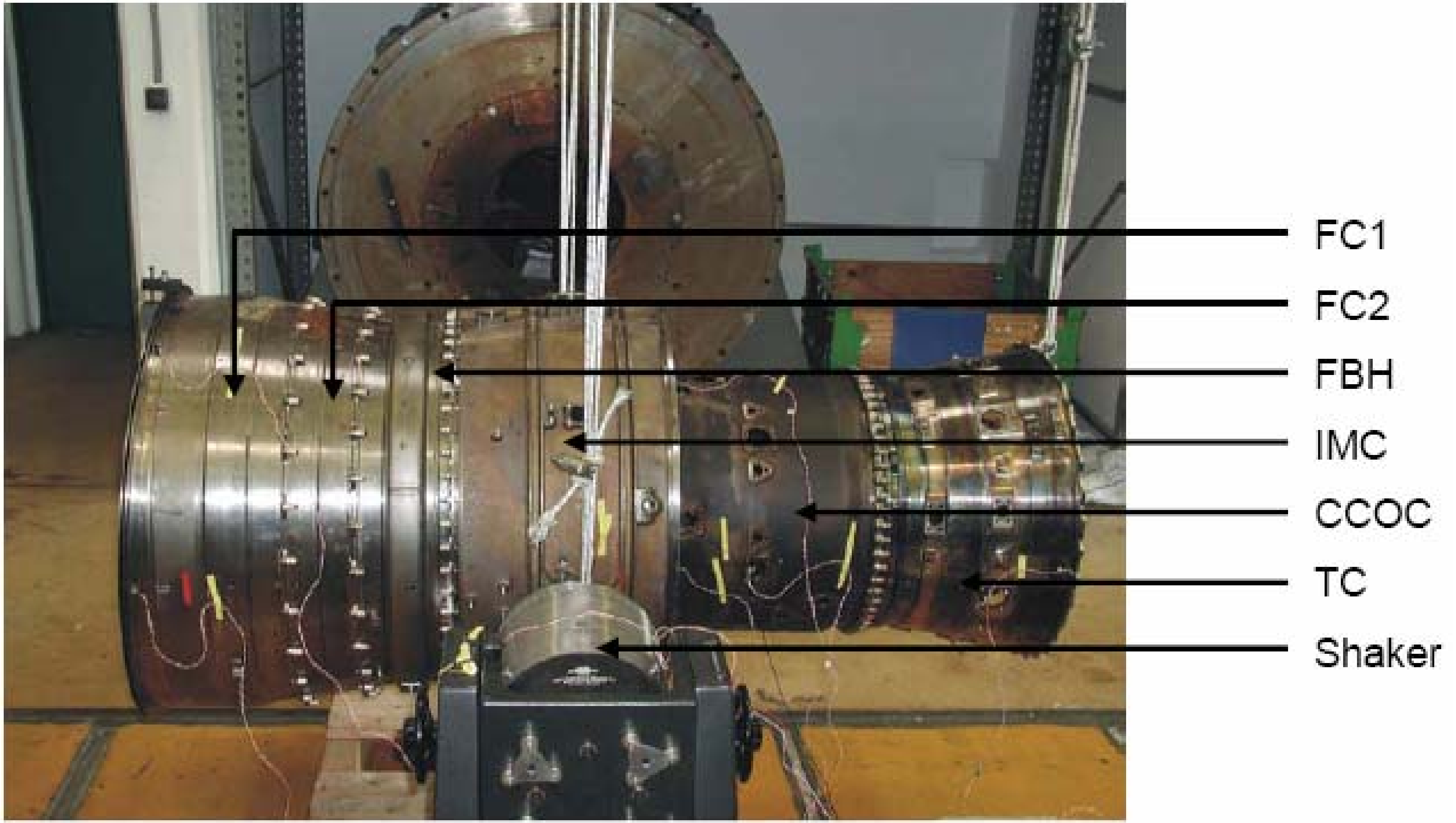
Softening-stiffness effect



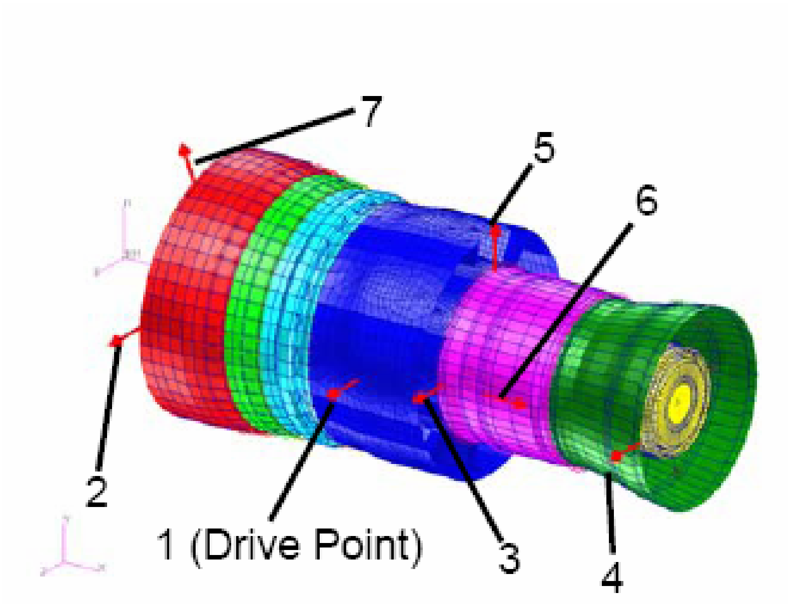


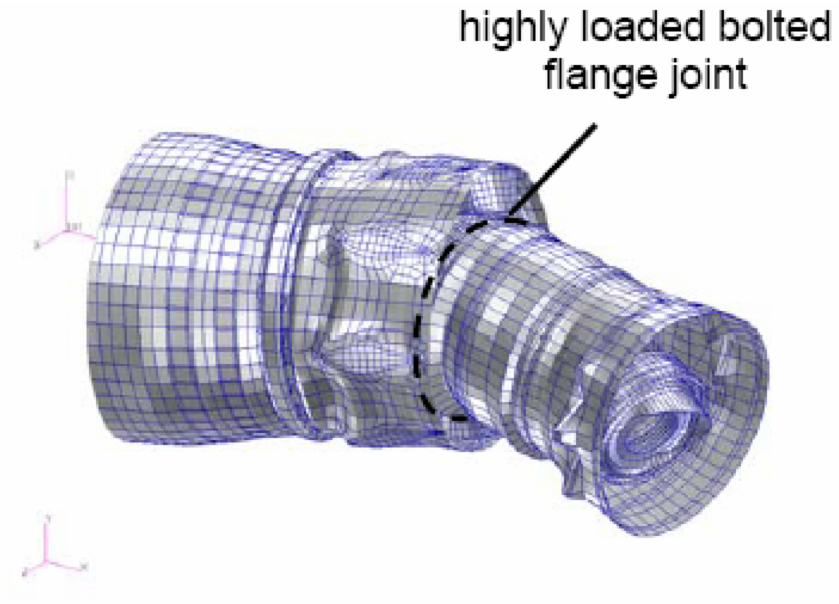


Softening-stiffness effect

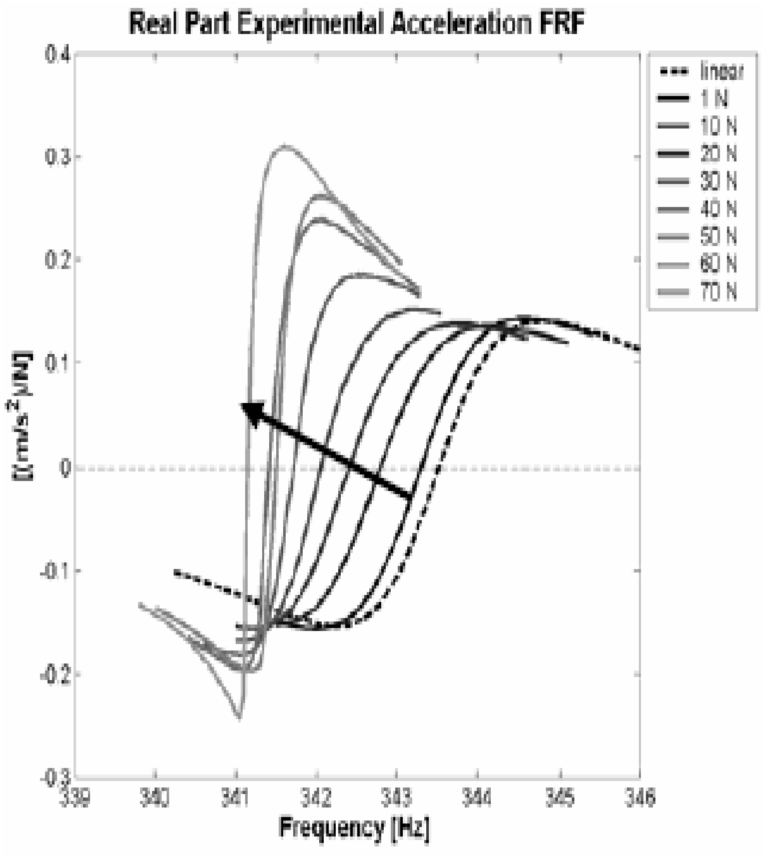


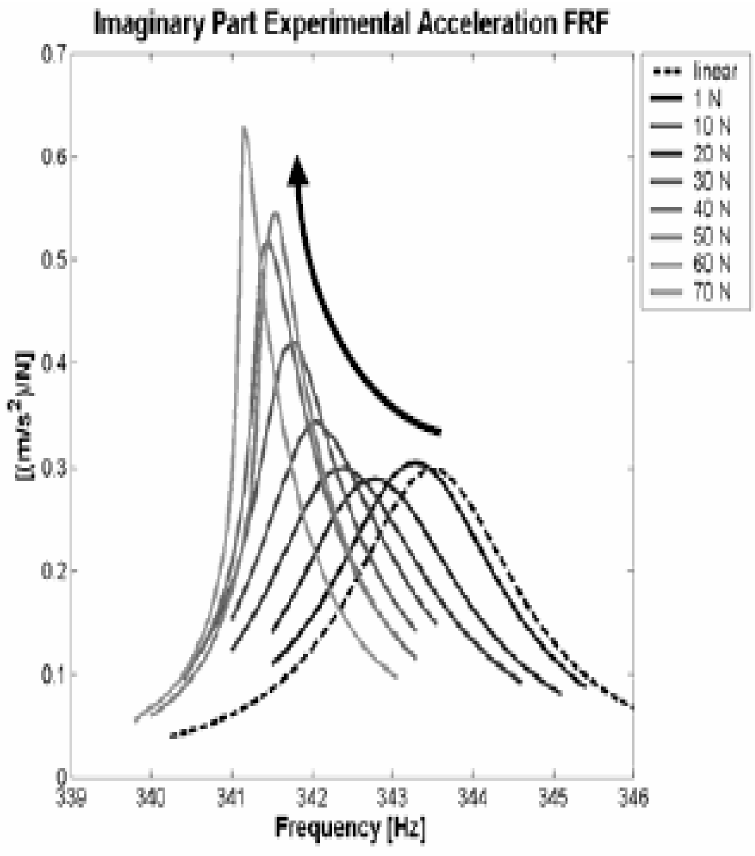
Softening-stiffness effect





Softening-stiffness effect





Coulomb friction nonlinearity

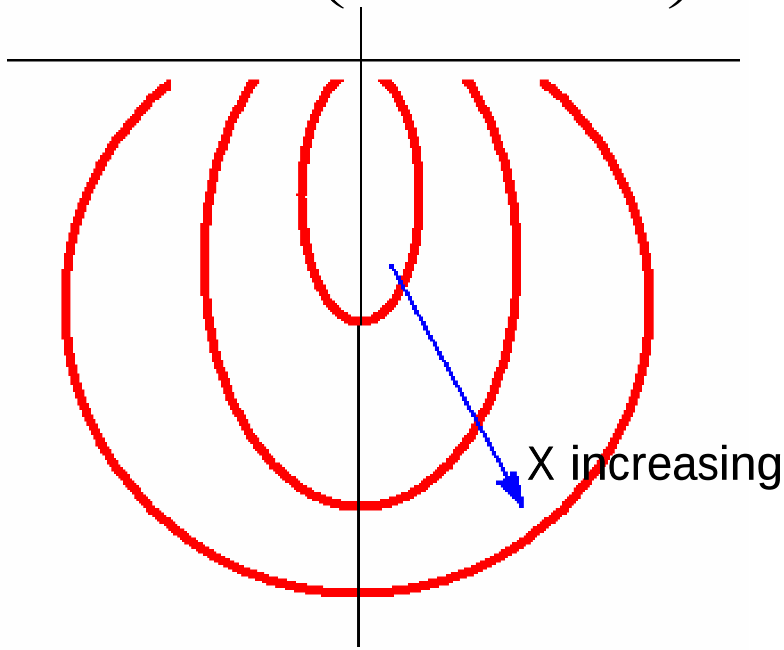
$$f_d(t) = c\dot{x}(t) + c_F \frac{\dot{x}(t)}{|\dot{x}(t)|}$$

$$\Delta E = 4c_F X \Rightarrow c_{eq} = \frac{\Delta E}{\int_0^{2\pi/\omega} \dot{x}(t)^2\, dt} = \frac{4c_F}{\pi\omega X}$$

$$\frac{X}{F} = \frac{1}{k - m\omega^2 + i\left(c\omega + \frac{4c_F}{\pi\omega X}\right)}$$

Coulomb friction nonlinearity

$$\frac{X}{F}=\frac{1}{k-m\omega^2+i\left(c\omega+\frac{4c_F}{\pi\omega X}\right)}$$



Other nonlinearities and other descriptions

- Backlash
- Bilinear Stiffness
- Microslip friction damping
- Quadratic (and other power law damping)
- .....

Modal Testing

(Lecture 2)

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MODAL ANALYSIS THEORY

- Understanding of how the structural parameters of mass, damping, and stiffness relate to
  - the impulse response function (time domain),
  - the frequency response function (Fourier, or frequency domain), and
  - the transfer function (Laplace domain)
- for single and multiple degree of freedom systems.

Theoretical Basis

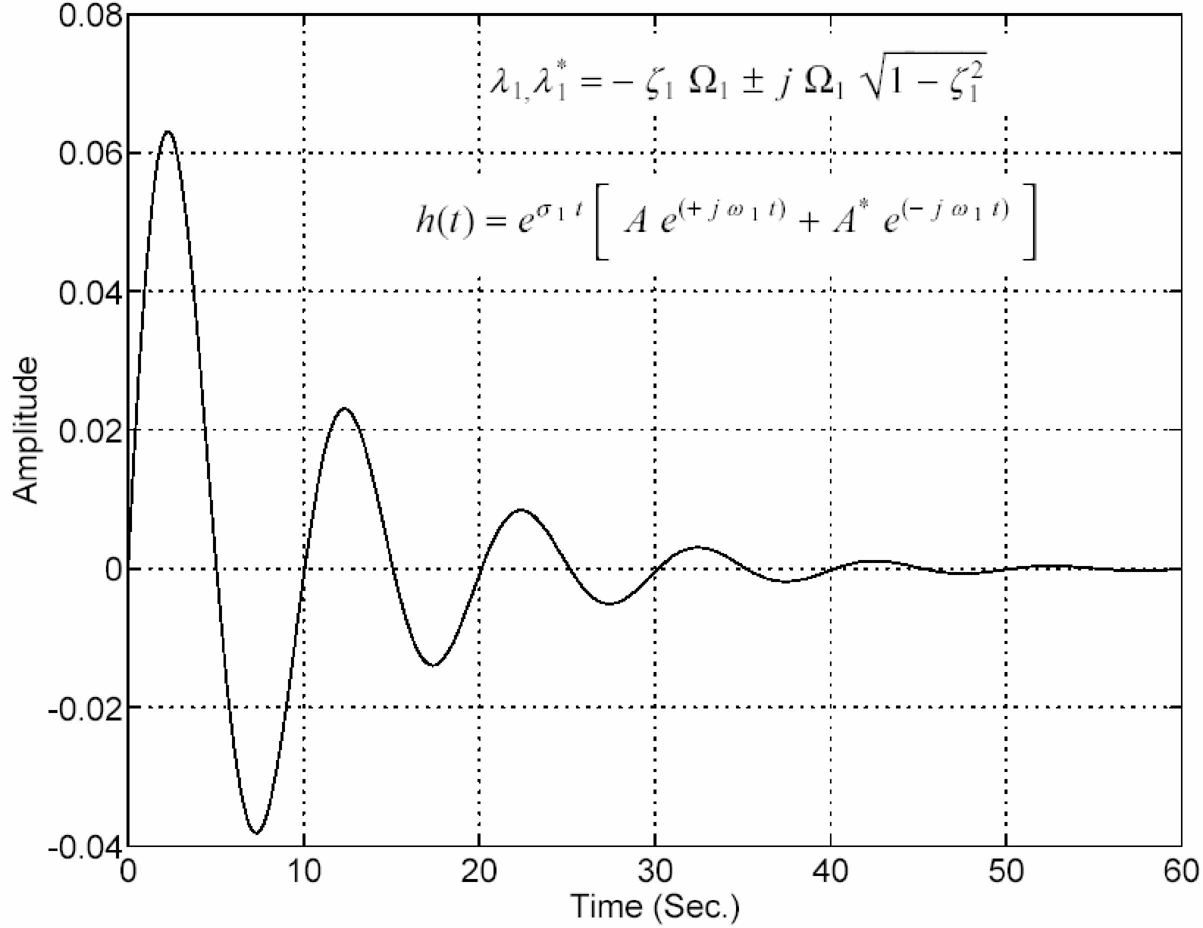
- SDOF system
  - Time Domain: Impulse Response Function
  - Presentation of FRF
  - Properties of FRF
- Undamped MDOF system
- MDOF system with proportional damping

SDOF System

- Three classes of system:
  - Undamped
  - Viscously-damped
  - Structurally Damped
- Response Models:

$$H(\omega)=\frac{X(\omega)}{F(\omega)}=\begin{cases}\dfrac{1}{k-m\omega^2}\\ \dfrac{1}{k-m\omega^2+ic\omega}\\ \dfrac{1}{k-m\omega^2+id}\end{cases}$$

Time Domain: Impulse Response Function



Frequency Domain: Frequency Response Function

$$\left[-M\,\omega^2 + j\,C\,\omega + K \right] X(\omega) = F(\omega) \qquad H(\omega) = \frac{X(\omega)}{F(\omega)}$$

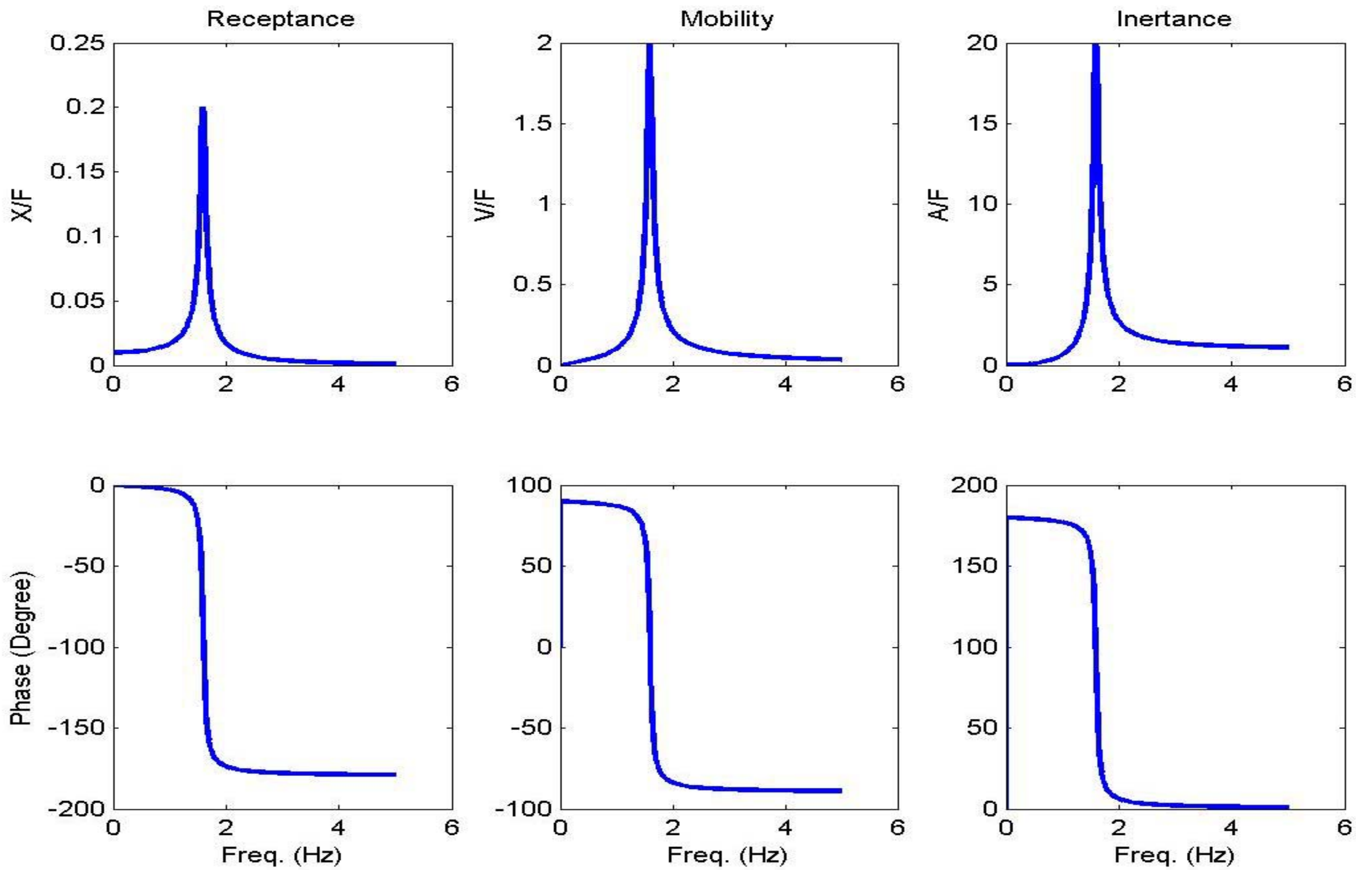
$$H(\omega) = \frac{1}{-M\,\omega^2 + j\,C\,\omega + K} = \frac{1/M}{-\omega^2 + j\left(\frac{C}{M}\right)\omega + \left(\frac{K}{M}\right)}$$

$$H(\omega) = \frac{1/M}{(j\,\omega - \lambda_1)(j\,\omega - \lambda_1^*)} = \frac{A}{(j\,\omega - \lambda_1)} + \frac{A^*}{(j\,\omega - \lambda_1^*)}$$

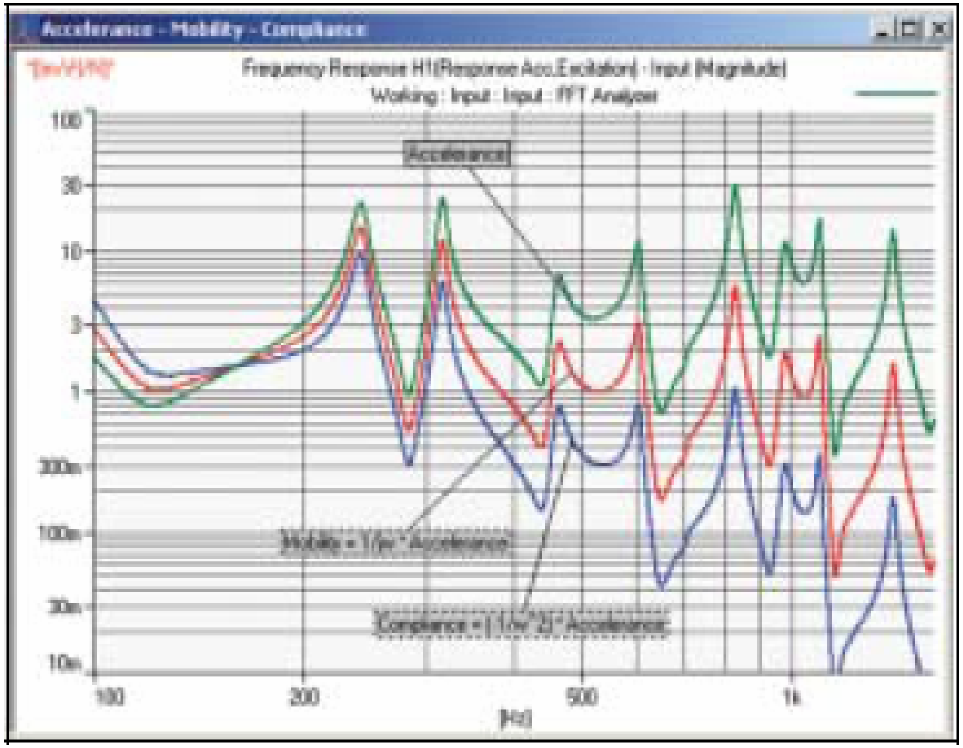
Alternative Forms of FRF

- Receptance $\dfrac{X(\omega)}{F(\omega)}$
  - Inverse is "Dynamic Stiffness"
- Mobility $\dfrac{V(\omega)}{F(\omega)} = i\omega \dfrac{X(\omega)}{F(\omega)}$
  - Inverse is "Dynamic Impedance"
- Inertance $\dfrac{A(\omega)}{F(\omega)} = -\omega^2 \dfrac{X(\omega)}{F(\omega)}$
  - Inverse is "Apparent mass"

Graphical Display of FRF

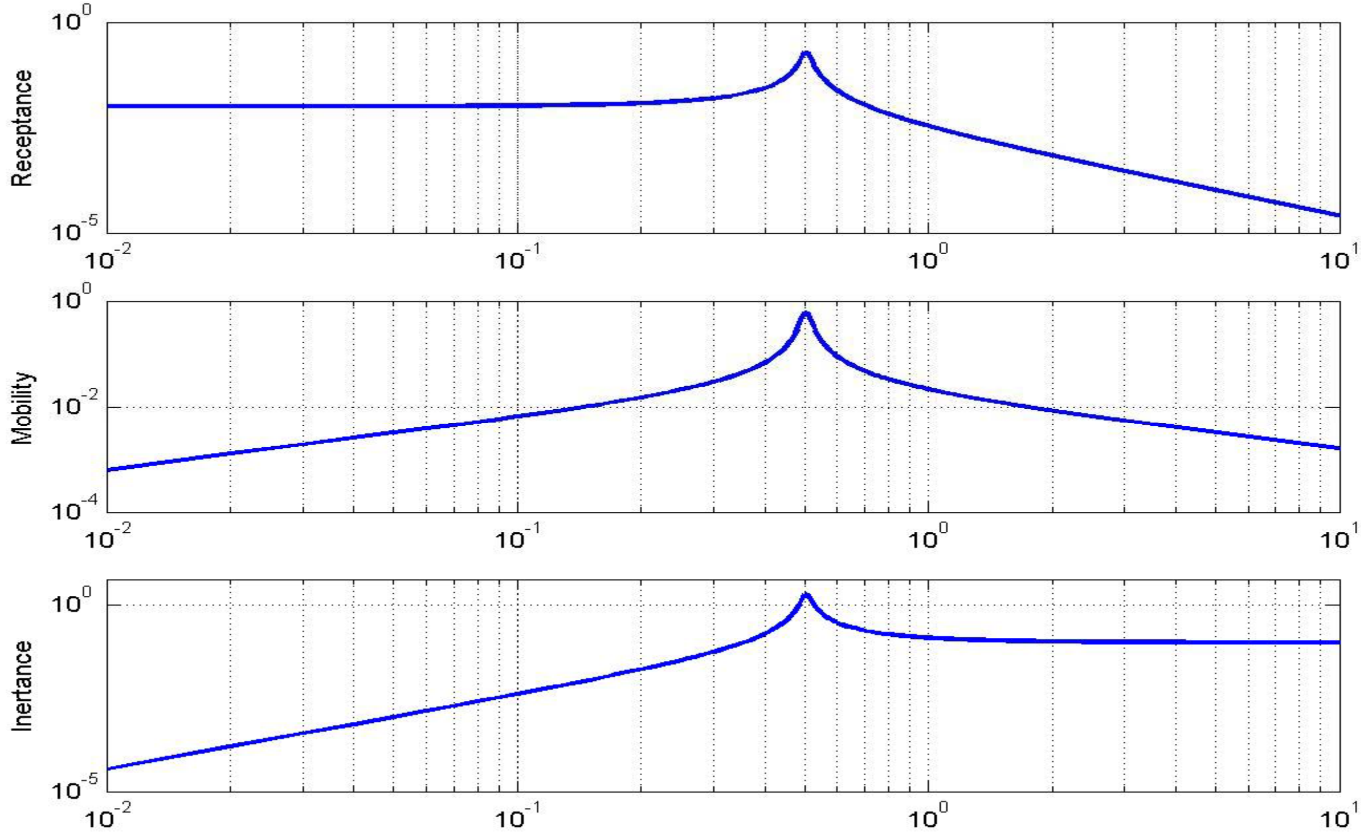


Graphical Display of FRF



The magnitude of the three mobility functions (accelerance, mobility and compliance)

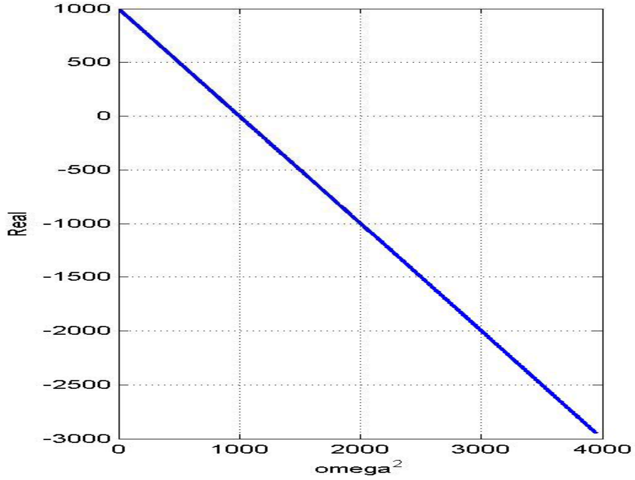
Stiffness and Mass Lines

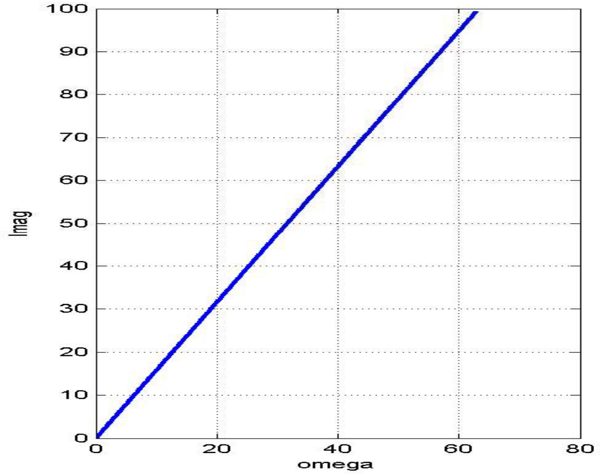


Reciprocal Plots

- The "inverse" or "reciprocal" plots
  - Real part
  - Imaginary part

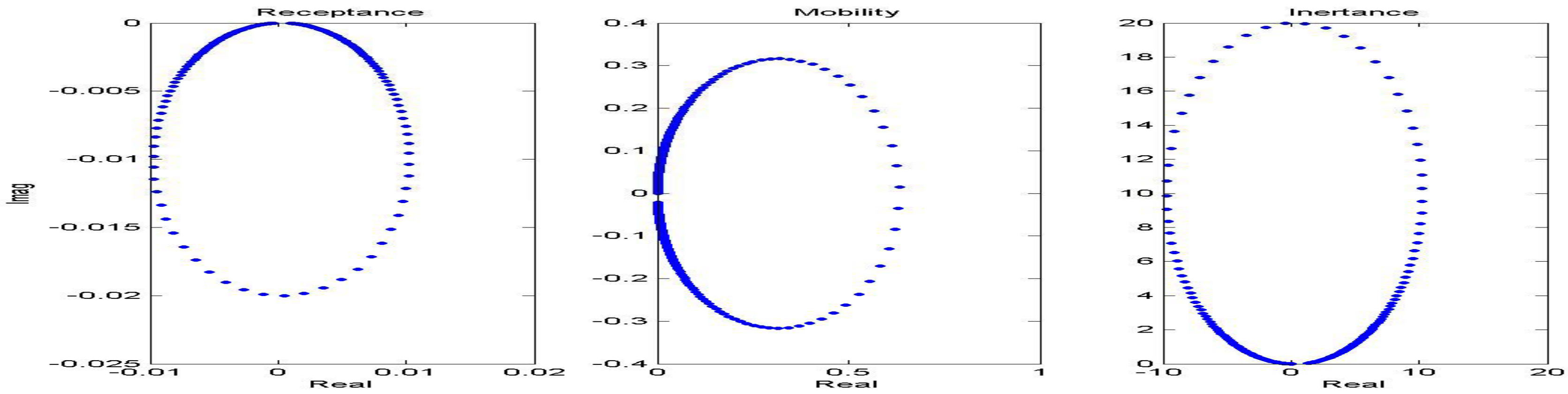
$$\Rightarrow \begin{cases} \mathrm{Re}\left(\dfrac{F(\omega)}{X(\omega)}\right) = k - m\omega^2 \\ \mathrm{Im}\left(\dfrac{F(\omega)}{X(\omega)}\right) = c\omega \end{cases}$$





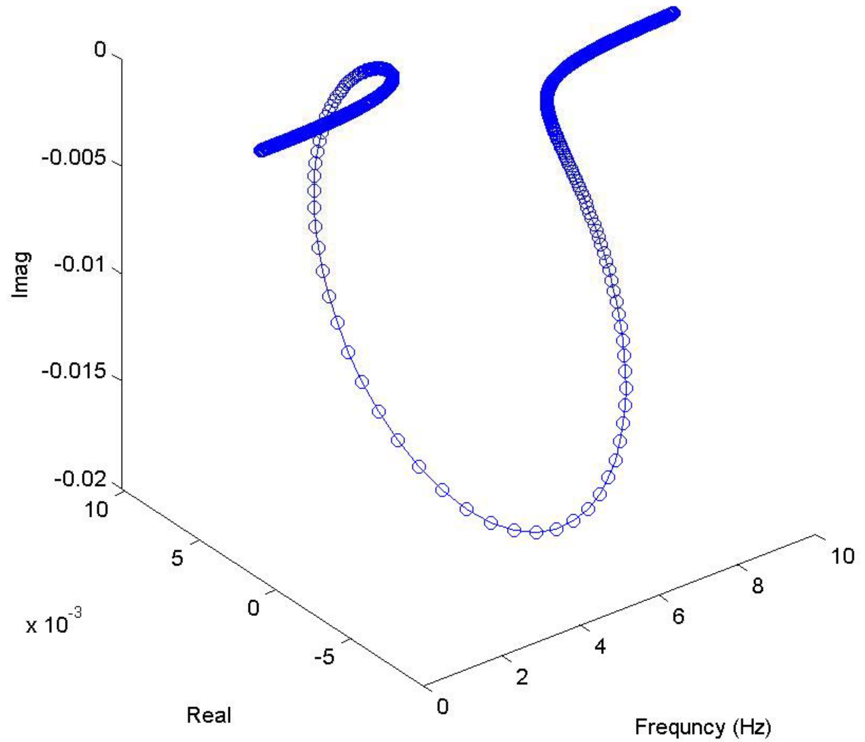
Nyquist Plot

- For viscous damping the Mobility plot is a circle.



- For structural damping the Receptance and Inertance plots are circles.

3D FRF Plot (SDOF)



Properties of SDOF FRF Plots

- Nyquist Mobility for viscose damping

$$Y(\omega) = \frac{i\omega}{k - m\omega^2 + ic\omega}$$

$$\mathrm{Re}(Y) = \frac{c\omega^2}{(k - m\omega^2)^2 + (c\omega)^2} \qquad \mathrm{Im}(Y) = \frac{\omega(k - m\omega^2)}{(k - m\omega^2)^2 + (c\omega)^2}$$

$$U = \left(\mathrm{Re}(Y) - \frac{1}{2c}\right), \quad V = \mathrm{Im}(Y)$$

$$U^2 + V^2 = \frac{\left((k - m\omega^2)^2 + (c\omega)^2\right)^2}{4c^2\left((k - m\omega^2)^2 + (c\omega)^2\right)^2} = \left(\frac{1}{2c}\right)^2$$

Properties of SDOF FRF Plots

- Nyquist Receptance for structural damping

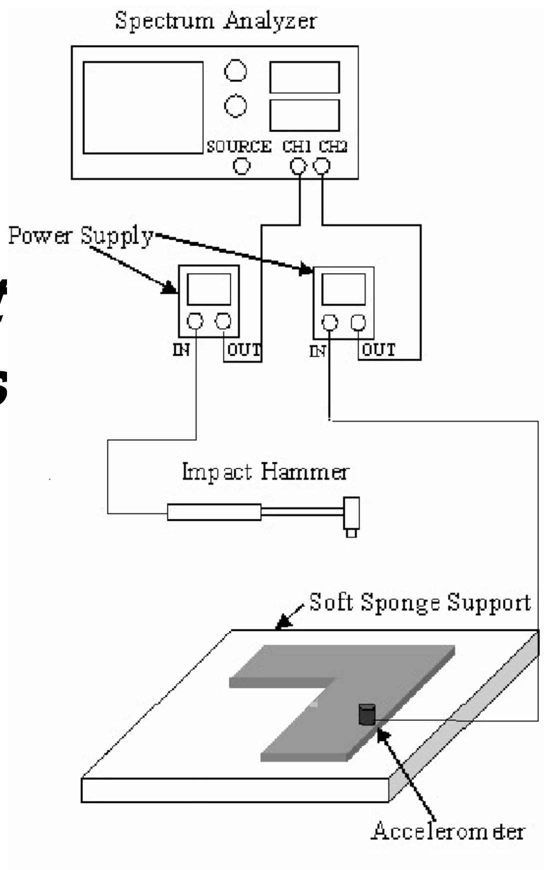
$$H(\omega)=\frac{1}{k+id-m\omega^2}=\frac{\left(k-m\omega^2\right)-id}{\left(k-m\omega^2\right)^2+d^2}$$

$$U=\frac{\left(k-m\omega^2\right)}{\left(k-m\omega^2\right)^2+d^2},V=\frac{d}{\left(k-m\omega^2\right)^2+d^2}$$

$$U^2+\left(V+\frac{1}{2d}\right)^2=\left(\frac{1}{2d}\right)^2$$

A Demo

- **Basic Assumptions**
  - ***The structure is assumed to be linear***
  - ***The structure is time invariant***
  - ***The structure obeys Maxwell's reciprocity***
  - ***The structure is observable***
    - loose components, or degrees-of-freedom of motion that are not measured, are not completely observable.



Modal Testing

(Lecture 3)

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Theoretical Basis

- Undamped MDOF Systems
- MDOF Systems with Proportional Damping
- MDOF Systems with General Structural Damping
- General Force Vector
- Undamped Normal Mode

Undamped MDOF Systems

- The equation of motion:

$$[M]\{\ddot{x}(t)\}+[K]\{x(t)\}=\{f(t)\}$$

- The modal model: $[\Phi],\Gamma=diag(\omega_1^2,\omega_2^2,\ldots,\omega_N^2)$
- The orthogonality:

$$[\Phi]^T[M][\Phi]=[I],[\Phi]^T[K][\Phi]=[\Gamma].$$

- Forced response solution:

$$([K]-\omega^2[M])\{X\}e^{i\omega t}=\{F\}e^{i\omega t}$$

$$\{X\}=([K]-\omega^2[M])^{-1}\{F\}\Rightarrow\{X\}=[\alpha(\omega)]\{F\}$$

Undamped MDOF Systems
(continued)

- Response Model

$$\left([K]-\omega^2[M]\right)=[\alpha(\omega)]^{-1}$$

$$[\Phi]^T\left([K]-\omega^2[M]\right)[\Phi]=[\Phi]^T[\alpha(\omega)]^{-1}[\Phi]$$

$$\left([\Gamma]-\omega^2[I]\right)=[\Phi]^T[\alpha(\omega)]^{-1}[\Phi]$$

$$[\alpha(\omega)]^{-1}=[\Phi]^{-T}\left([\Gamma]-\omega^2[I]\right)[\Phi]^{-1}$$

$$[\alpha(\omega)]=[\Phi]\left([\Gamma]-\omega^2[I]\right)^{-1}[\Phi]^T$$

Undamped MDOF Systems (continued)

- The receptance matrix is symmetric.

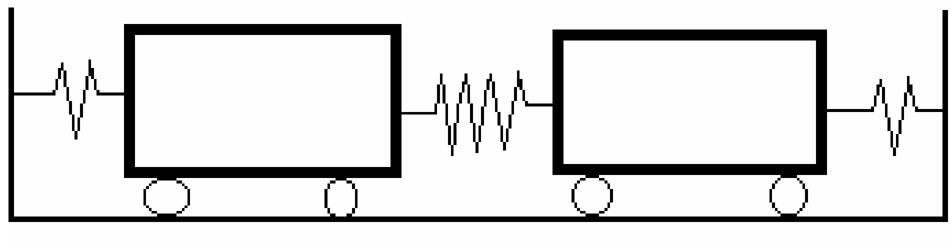
Single Input

$$\alpha_{jk} = \frac{X_j}{F_k} = \alpha_{kj} = \frac{X_k}{F_j},$$

Modal Constant / Modal Residue

$$\alpha_{jk}(\omega) = \sum_{r=1}^{N} \frac{\phi_{jr}\phi_{kr}}{\omega_r^2 - \omega^2} = \sum_{r=1}^{N} \frac{{}_rA_{jk}}{\omega_r^2 - \omega^2}$$

Example:

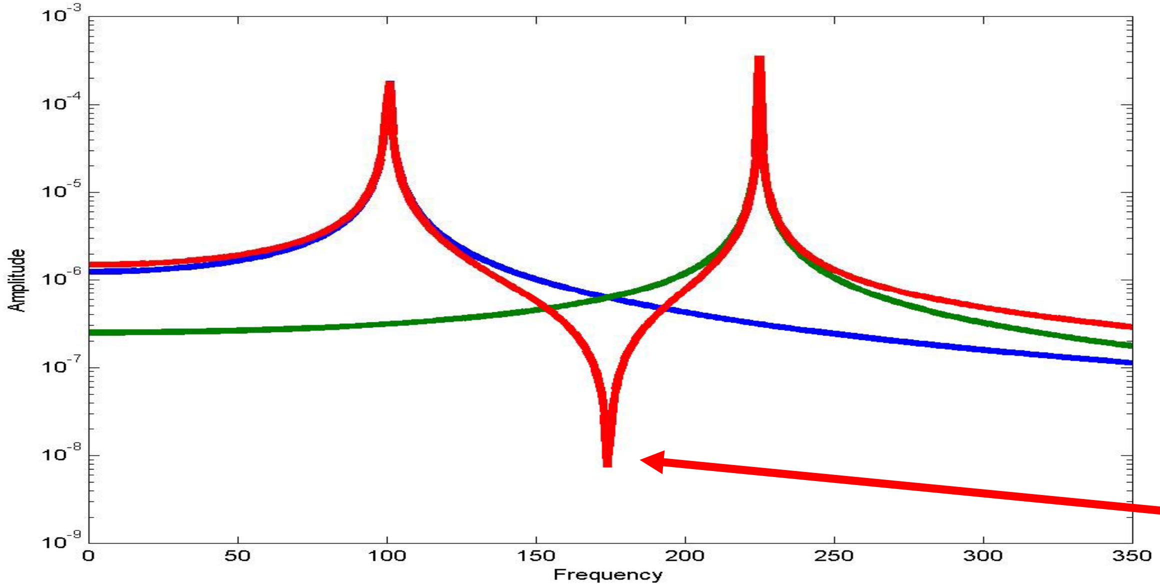


$$[M]=\begin{bmatrix}1 & \\ & 1\end{bmatrix}kg \qquad [K]=\begin{bmatrix}1.2 & -0.8\\ -0.8 & 1.2\end{bmatrix}MN/m$$

$$\left[\omega_r^2\right]=\begin{bmatrix}4e5 & \\ & 2e6\end{bmatrix} \qquad [\Phi]=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$\alpha_{11}(\omega)=\frac{0.5}{4e5-\omega^2}+\frac{05}{2e6-\omega^2}=\frac{1.2e6-\omega^2}{8e11-2.4e6\omega^2+\omega^4}.$$

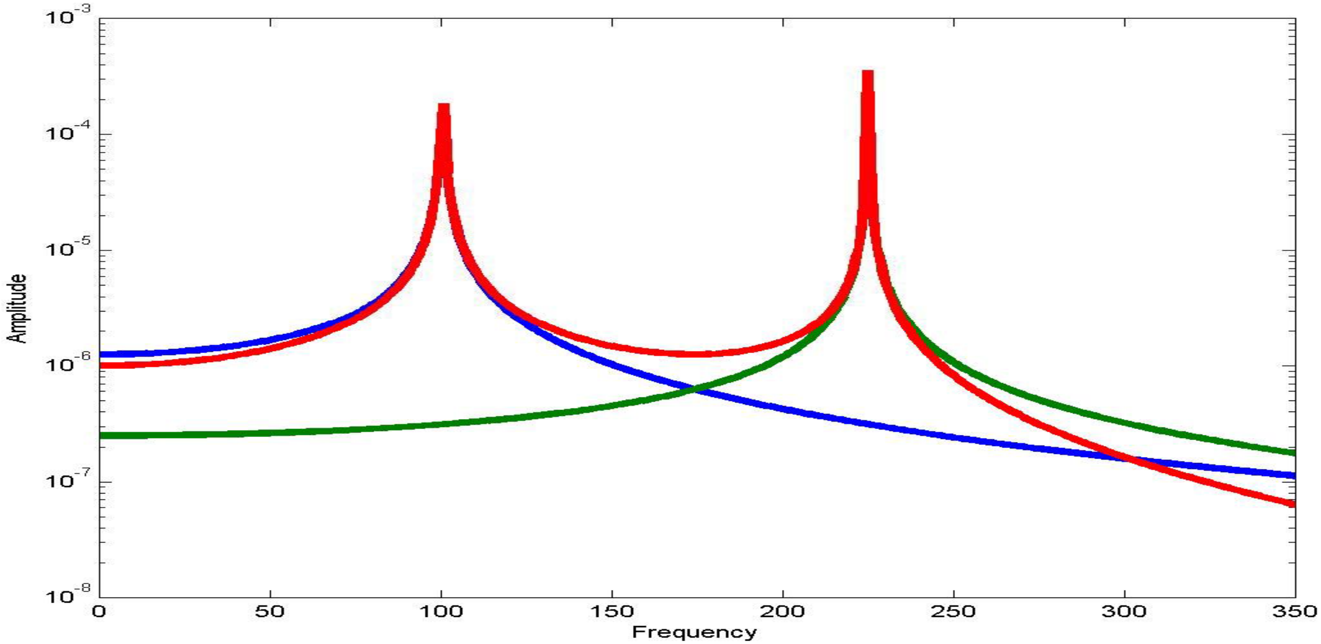
Example (continued):



$$\alpha_{11}(\omega) = \frac{0.5}{4e5-\omega^2} + \frac{05}{2e6-\omega^2} = \frac{1.2e6-\omega^2}{8e11-2.4e6\omega^2+\omega^4}.$$

Poles

Example (continued):



$$\alpha_{12}(\omega) = \frac{0.5}{4e5 - \omega^2} - \frac{05}{2e6 - \omega^2} = \frac{8e4}{8e11 - 2.4e6\omega^2 + \omega^4}.$$

MDOF Systems with Proportional Damping

- A proportionally damped matrix is diagonalized by normal modes of the corresponding undamped system

$$[\Phi]^T[D][\Phi] = diag(d_1, d_2, \cdots, d_N)$$

  - Special cases:

$$[D] = \beta[K],$$

$$[D] = \delta[M],$$

$$[D] = \beta[K] + \delta[M].$$

MDOF Systems with Structurally Proportional Damping

- Response Model

$$\left([K]+i[D]-\omega^2[M]\right)=[\alpha(\omega)]^{-1}$$

$$[\Phi]^T\left([K]+i[D]-\omega^2[M]\right)[\Phi]=[\Phi]^T[\alpha(\omega)]^{-1}[\Phi]$$

$$\left(\left[\omega_r^2(1+i\eta_r^2)\right]-\omega^2[I]\right)=[\Phi]^T[\alpha(\omega)]^{-1}[\Phi]$$

$$[\alpha(\omega)]^{-1}=[\Phi]^{-T}\left(\left[\omega_r^2(1+i\eta_r^2)\right]-\omega^2[I]\right)[\Phi]^{-1}$$

$$[\alpha(\omega)]=[\Phi]\left(\left[\omega_r^2(1+i\eta_r^2)\right]-\omega^2[I]\right)^{-1}[\Phi]^T$$

$$\alpha_{jk}(\omega)=\sum_{r=1}^{N}\frac{\phi_{jr}\phi_{kr}}{\omega_r^2(1+i\eta_r^2)-\omega^2}$$

Real Residue (numerator)

Complex Pole (denominator)

MDOF Systems with Viscously Proportional Damping

- Response Model

$$\left([K]+i\omega[C]-\omega^2[M]\right)=[\alpha(\omega)]^{-1}$$

$$[\Phi]^T\left([K]+i\omega[C]-\omega^2[M]\right)[\Phi]=[\Phi]^T[\alpha(\omega)]^{-1}[\Phi]$$

$$\left(\left[\omega_r^2\right]+i\omega[2\zeta_r\omega_r]-\omega^2[I]\right)=[\Phi]^T[\alpha(\omega)]^{-1}[\Phi]$$

$$[\alpha(\omega)]^{-1}=[\Phi]^{-T}\left(\left[\omega_r^2\right]+i\omega[2\zeta_r\omega_r]-\omega^2[I]\right)[\Phi]^{-1}$$

$$[\alpha(\omega)]=[\Phi]\left(\left[\omega_r^2\right]+i\omega[2\zeta_r\omega_r]-\omega^2[I]\right)^{-1}[\Phi]^T$$

$$\alpha_{jk}(\omega)=\sum_{r=1}^{N}\frac{\phi_{jr}\phi_{kr}}{\omega_r^2-\omega^2+2\zeta_r\omega_r\omega}$$

MDOF Systems with General Structural Damping

- The equation of motion:

$$[M]\{\ddot{x}(t)\} + ([K] + i[D])\{x(t)\} = \{f(t)\}$$

- The orthogonality:

$$[\Phi]^T[M][\Phi] = [I], [\Phi]^T[K + iD][\Phi] = [\Gamma].$$

Complex Mode Shapes

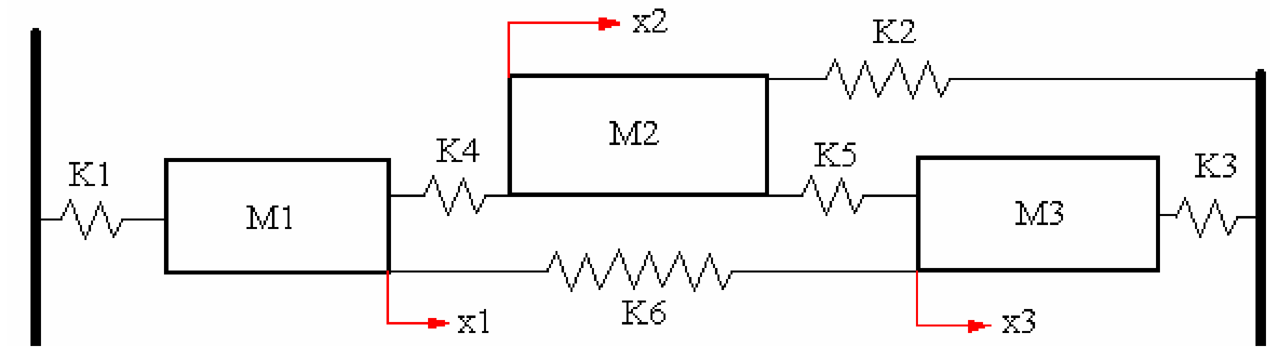
Complex Eigen-values

- Forced response solution:

$$([K] + i[D] - \omega^2[M])\{X\}e^{i\omega t} = \{F\}e^{i\omega t}$$

$$\{X\} = ([K] + i[D] - \omega^2[M])^{-1}\{F\} \Rightarrow \{X\} = [\alpha(\omega)]\{F\}$$

Example:



$$Model \quad 1$$

$$m_1 = 0.5kg, m_2 = 1kg, m_3 = 1.5kg$$

$$k_j = 1e3N/m, \; j = 1,\ldots,6$$

$$Undamped$$

$$\Gamma = \begin{bmatrix} 950 & & \\ & 3352 & \\ & & 6698 \end{bmatrix}, [\Phi] = \begin{bmatrix} 0.464 & -0.218 & -1.318 \\ 0.536 & -0.782 & 0.318 \\ 0.635 & 0.493 & 0.142 \end{bmatrix}$$

Example:

Proportional

$$[D] = 0.05[K]$$

$$\Gamma = (1 + i0.05)\begin{bmatrix} 950 & & \\ & 3352 & \\ & & 6698 \end{bmatrix}, [\Phi] = \begin{bmatrix} 0.464 & -0.218 & -1.318 \\ 0.536 & -0.782 & 0.318 \\ 0.635 & 0.493 & 0.142 \end{bmatrix}$$

Non-Proportional

$$d_1 = 0.3k_1, d_j = 0.0, j = 2, \ldots, 6$$

$$\Gamma = \begin{bmatrix} 957(1 + i0.067) & & \\ & 3354(1 + i0.042) & \\ & & 6690(1 + i0.078) \end{bmatrix},$$

$$[\Phi] = \begin{bmatrix} 0.463(-5.5^\circ) & 0.217(173^\circ) & 1.318(181^\circ) \\ 0.537(0.0^\circ) & 0.784(181^\circ) & 0.318(-6.7^\circ) \\ 0.636(1.0^\circ) & 0.492(-1.3^\circ) & 0.142(-3.1^\circ) \end{bmatrix}$$

Almost real modes

Example:

Model 2

$$m_1 = 1kg, m_2 = 0.95kg, m_3 = 1.05kg$$

$$k_j = 1e3N/m,\ j = 1,\ldots,6$$

Undamped

$$\Gamma = \begin{bmatrix} 999 & & \\ & 3892 & \\ & & 4124 \end{bmatrix}, [\Phi] = \begin{bmatrix} 0.577 & -0.602 & 0.552 \\ 0.567 & -0.215 & -0.827 \\ 0.587 & 0.752 & 0.207 \end{bmatrix}$$

Proportional

$$[D] = 0.05[K],$$

$$\Gamma = (1 + i0.05)\begin{bmatrix} 999 & & \\ & 3892 & \\ & & 4124 \end{bmatrix}, [\Phi] = \begin{bmatrix} 0.577 & -0.602 & 0.552 \\ 0.567 & -0.215 & -0.827 \\ 0.587 & 0.752 & 0.207 \end{bmatrix}$$

Example:

$$Non - \Pr oportional$$

$$d_1 = 0.3k_1, d_j = 0.0, \; j = 2,\ldots,6$$

$$\Gamma = \begin{bmatrix} 1006(1+i0.1) & & \\ & 3942(1+i0.031) & \\ & & 4067(1+i0.019) \end{bmatrix},$$

$$[\Phi] = \begin{bmatrix} 0.578(-4^\circ) & 0.851(162^\circ) & 0.685(40^\circ) \\ 0.569(2^\circ) & 0.570(101^\circ) & 1.019(176^\circ) \\ 0.588(2^\circ) & 0.848(12^\circ) & 0.560(-50^\circ) \end{bmatrix}$$

Heavily complex modes

MDOF Systems with General Structural Damping

$$\left([K]+i[D]-\omega^2[M]\right)=[\alpha(\omega)]^{-1}$$

$$[\Phi]^T\left([K]+i[D]-\omega^2[M]\right)[\Phi]=[\Phi]^T[\alpha(\omega)]^{-1}[\Phi]$$

$$\left(\left[\omega_r^2(1+i\eta_r^2)\right]-\omega^2[I]\right)=[\Phi]^T[\alpha(\omega)]^{-1}[\Phi]$$

$$[\alpha(\omega)]^{-1}=[\Phi]^{-T}\left(\left[\omega_r^2(1+i\eta_r^2)\right]-\omega^2[I]\right)[\Phi]^{-1}$$

$$[\alpha(\omega)]=[\Phi]\left(\left[\omega_r^2(1+i\eta_r^2)\right]-\omega^2[I]\right)^{-1}[\Phi]^T$$

$$\alpha_{jk}(\omega)=\sum_{r=1}^{N}\frac{\phi_{jr}\phi_{kr}}{\omega_r^2(1+i\eta_r^2)-\omega^2}$$

Complex Residues (numerator)

Complex Poles (denominator)

General Force Vector

- In many situations the system is excited at several points.

General Force Vector (continued)

- The response is governed by:

$$\left(\left[K+iD\right]-\omega^2\left[M\right]\right)\left\{X\right\}e^{i\omega t}=\left\{F\right\}e^{i\omega t}$$

- The solution:

$$\left\{X\right\}=\sum_{r=1}^{N}\frac{\left\{\phi\right\}_r^T\left\{F\right\}\left\{\phi\right\}_r}{\omega_r^2\left(1+i\eta_r^2\right)-\omega^2}$$

- All forces have the same frequency but may vary in magnitude and phase.

General Force Vector (continued)

- The response vector is referred to:
  - Forced Vibration Mode
  - or Operating Deflection Shape (ODS)
- When the excitation frequency is close to the natural frequency:
  - ODS reflects the shape of nearby mode
  - But not identical due to contributions of other modes.

General Force Vector (continued)

- Damped system normal mode:
  - By carefully tuning the force vector the response can be controlled by a single mode.
  - The is attained if $\{\phi\}_r^T \{F\}_s = \delta_{rs}$
  - Depending upon damping condition the force vector entries may well be complex (they have different phases)

Undamped Normal Mode

- Special Case of interest:
  - Harmonic excitation of mono-phased forces
    - Same frequency
    - Same phase
    - Magnitudes may vary
- Is it possible to obtain mono-phased response?

Undamped Normal Mode
(continued)

- The real force response amplitudes:

$$\{f(t)\} = \{\hat{F}\}e^{i\omega t}$$

$$\{x(t)\} = \{\hat{X}\}e^{i(\omega t-\theta)}$$

$$([K+iD]-\omega^2[M])\{\hat{X}\}e^{i\omega t} = \{\hat{F}\}e^{i\omega t}$$

- Real and imaginary parts:

$$(([K]-\omega^2[M])\cos\theta+[D]\sin\theta)\{\hat{X}\}=\{\hat{F}\}$$

$$(([K]-\omega^2[M])\sin\theta+[D]\cos\theta)\{\hat{X}\}=\{0\}$$

- The 2nd equation is an eigen-value problem; its solutions leads to real $\{\hat{F}\}$

N solutions

Undamped Normal Mode
(continued)

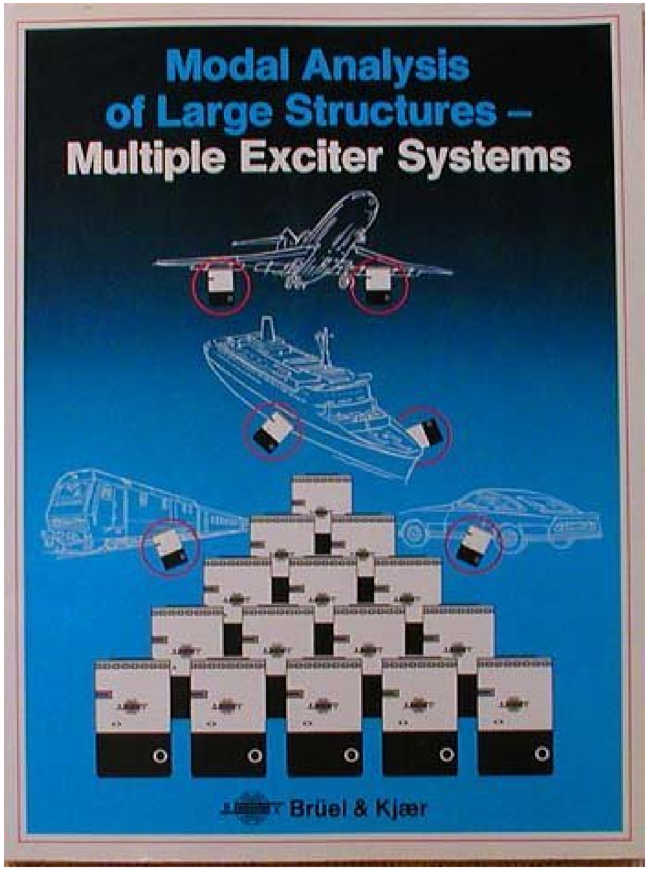
- At a frequency that the phase lag between all forces and all responses is 90 degree then

$$\left(\left([K]-\omega^2[M]\right)\sin\theta+[D]\cos\theta\right)\left\{\hat{X}\right\}=\{0\}$$

- Results
  - Undamped normal modes
  - Natural frequencies of undamped system

Undamped Normal Mode
(continued)

- The base for multi-shaker test procedures.
- Modal Analysis of Large Structures: Multiple Exciter Systems By: M. Phil. K. Zaveri



Modal Testing

(Lecture 4)

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Theoretical Basis

- General Force Vector
- Undamped Normal Mode
- MDOF System with General Viscous Damping
- Force Response Solution/ General Viscous Damping

General Force Vector

- In many situations the system is excited at several points.

General Force Vector

- Otherwise you end up damaging the structure!!!!

General Force Vector (continued)

- The response is governed by:

$$\left(\left[K+iD\right]-\omega^2\left[M\right]\right)\left\{X\right\}e^{i\omega t}=\left\{F\right\}e^{i\omega t}$$

- All forces have the same frequency but may vary in magnitude and phase.
- The solution:

$$\left\{X\right\}=\sum_{r=1}^{N}\frac{\left\{\phi\right\}_r^T\left\{F\right\}\left\{\phi\right\}_r}{\omega_r^2\left(1+i\eta_r^2\right)-\omega^2}$$

General Force Vector (continued)

- The response vector is referred to:
  - Forced Vibration Mode
  - or Operating Deflection Shape (ODS)
- When the excitation frequency is close to the natural frequency:
  - ODS reflects the shape of nearby mode
  - But not identical due to contributions of other modes.

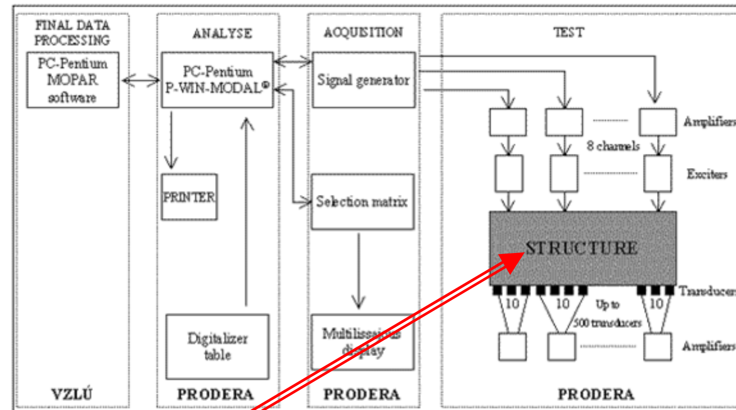
General Force Vector (continued)

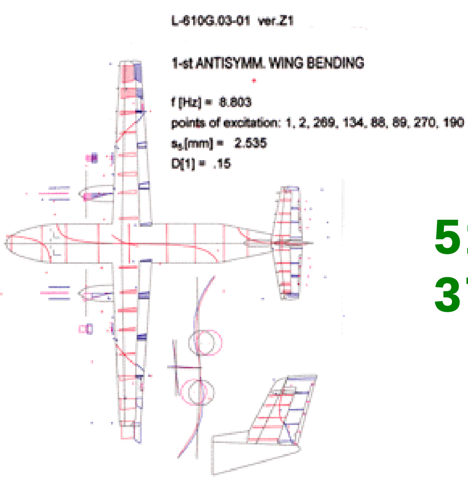
- Damped system normal mode:
  - By carefully tuning the force vector the response can be controlled by a single mode.
  - This is attained if $\{\phi\}_r^T \{F\}_s = \delta_{rs}$
  - Depending upon damping condition the force vector entries may well be complex (they have different phases)

Undamped Normal Mode

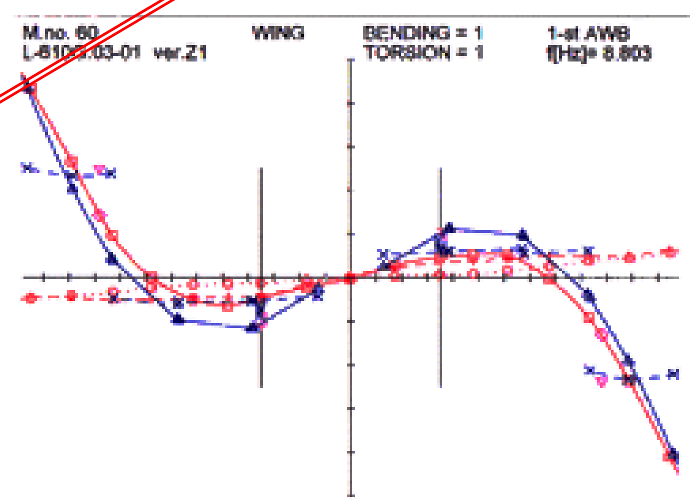
- Special Case of interest:
  - Harmonic excitation of mono-phased forces
    - Same frequency
    - Same phase
    - Magnitudes may vary
- Is it possible to obtain mono-phased response?

Undamped Normal Mode





512 channel
37 Shakers



Undamped Normal Mode
(continued)

- The real force response amplitudes:

$$\{f(t)\} = \{\hat{F}\}e^{i\omega t}$$

$$\{x(t)\} = \{\hat{X}\}e^{i(\omega t-\theta)}$$

$$\left([K+iD]-\omega^2[M]\right)\{\hat{X}\}e^{i(\omega t-\theta)} = \{\hat{F}\}e^{i\omega t}$$

- Real and imaginary parts:

$$\left(\left([K]-\omega^2[M]\right)\cos\theta+[D]\sin\theta\right)\{\hat{X}\}=\{\hat{F}\}$$

$$\left(\left([K]-\omega^2[M]\right)\sin\theta+[D]\cos\theta\right)\{\hat{X}\}=\{0\}$$

- The 2nd equation is an eigen-value problem; its solutions leads to real $\{\hat{F}\}$

N solutions

Undamped Normal Mode
(continued)

- At a frequency that the phase lag between all forces and all responses is 90 degree then

$$\left(\left(\left[K\right]-\omega^2\left[M\right]\right)\sin\theta+\left[D\right]\cos\theta\right)\left\{\hat{X}\right\}=\left\{0\right\}$$

- Results $\Rightarrow\left(\left[K\right]-\omega^2\left[M\right]\right)\left\{\hat{X}\right\}=\left\{0\right\}$
  - Undamped normal modes
  - Natural frequencies of undamped system

Undamped Normal Mode
(continued)

- The base for multi-shaker test procedures.
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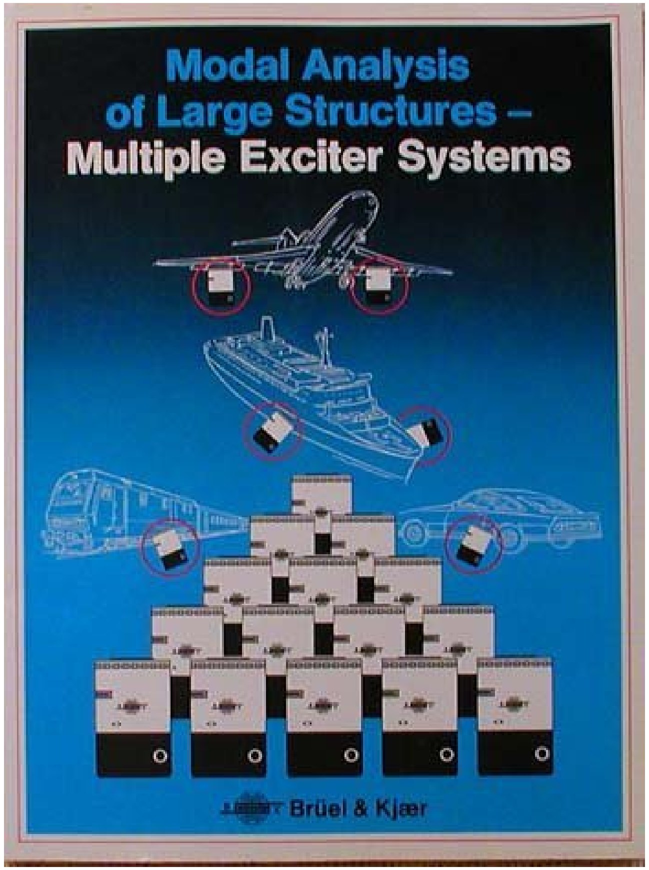

MDOF System with General Viscous Damping

$$E.O.M. \Rightarrow [M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f\}$$

$$\{f(t)\}=\{F\}e^{i\omega t} \Rightarrow \{x(t)\}=\{X\}e^{i\omega t}$$

$$\{X\}=\left([K]-\omega^2[M]+i\omega[C]\right)^{-1}\{F\}$$

- Next the orthogonality properties of the system in 2N space is used for force response solution.

Force Response Solution

$$EOM \Rightarrow \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix}$$

$$Free \quad Vib. \Rightarrow \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \{\dot{u}\} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \{u\} = \{0\}$$

$$Eigen-solution \Rightarrow \left(s_r \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \right) \{u_r\} = \{0\}$$

$$\Rightarrow U^T \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} U = I, U^T \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} U = diag(s_1, s_2, \cdots, s_{2N}).$$

Force Response Solution

$$\begin{Bmatrix} X \\ i\omega X \end{Bmatrix} = \sum_{r=1}^{2N} \frac{u_r^T \begin{Bmatrix} F \\ 0 \end{Bmatrix} u_r}{i\omega - s_r} = \sum_{r=1}^{N} \frac{u_r^T \begin{Bmatrix} F \\ 0 \end{Bmatrix} u_r}{i\omega - s_r} + \frac{u_r^H \begin{Bmatrix} F \\ 0 \end{Bmatrix} u_r^*}{i\omega - s_r^*}$$

- The above simplification is due to the fact that eigen-values and eigen-vectors occur in complex conjugate pairs.

Force Response Solution

- Single point excitation:

$$\alpha_{jk}(\omega) = \sum_{r=1}^{N} \frac{u_{jr}u_{kr}}{i\omega - s_r} + \frac{u_{jr}^{*}u_{kr}^{*}}{i\omega - s_r^{*}}$$

Modal Testing

(Lecture 5)

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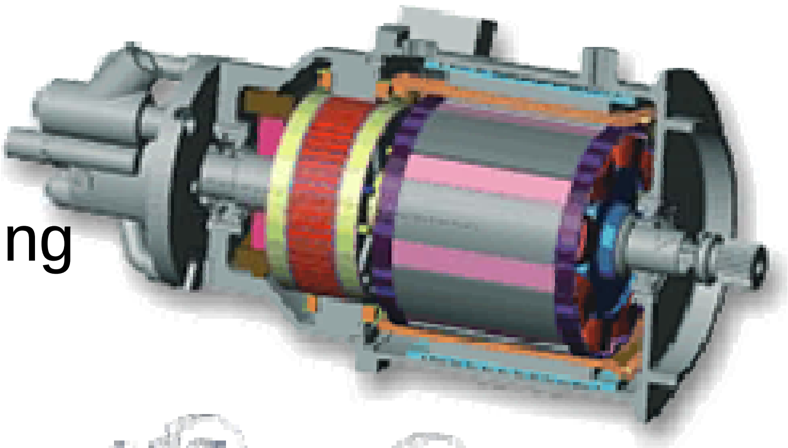
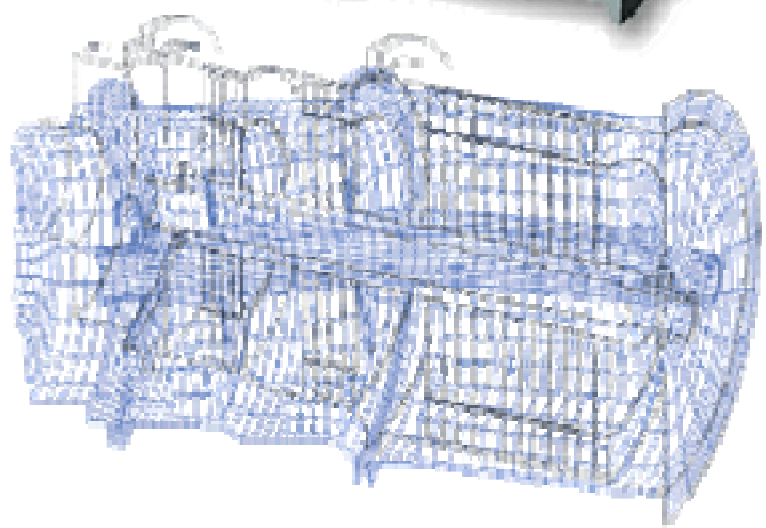
Modal Analysis of Rotating Structures

- Non-symmetry in system matrices
- Modes of undamped rotating system
  - Symmetric Stator
  - Non-Symmetric Stator
- FRF's of rotating system
- Out-of-balance excitation
  - Synchronous excitation
  - Non-Synchronous excitation



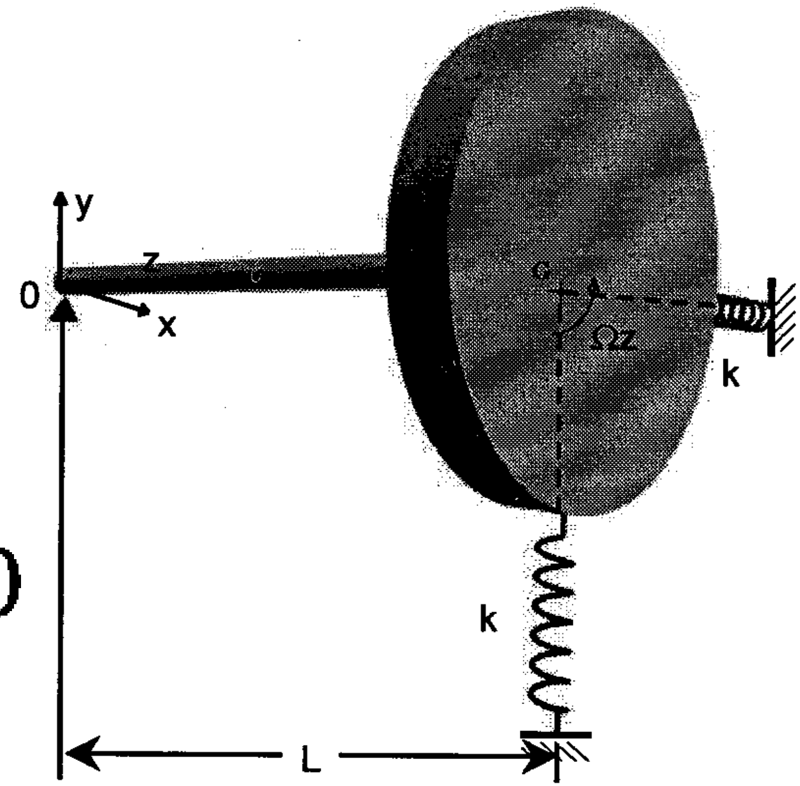


Non-symmetry in System Matrices

- The rotating structures are subject to additional forces:
  - Gyroscopic forces
  - Rotor-stator rub forces
  - Electrodynamic forces
  - Unsteady aerodynamic forces
  - Time varying fluid forces
- These forces can destroy the symmetry of the system matrices.

Non-rotating system properties

- A rigid disc mounted on the free end of a rigid shaft of length L,
- The other end of is effectively pin-jointed.

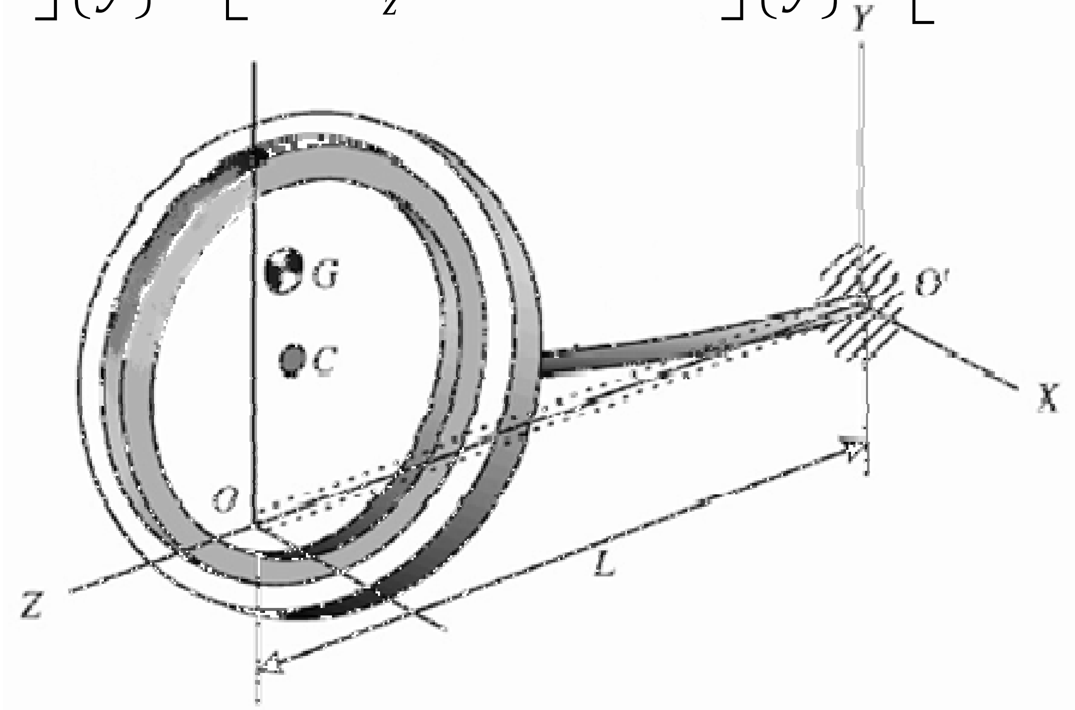


$$(I_0 / L)\ddot{x} + k_x L \; x \;\; = 0$$

$$(I_0 / L)\ddot{y} + k_y L \; y \;\; = 0$$

Modes of Undamped Rotating System

$$\begin{bmatrix} I_0/L & 0 \\ 0 & I_0/L \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} 0 & J\Omega_z/L \\ -J\Omega_z/L & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} k_x L & 0 \\ 0 & k_y L \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$



Symmetric stator

$$k_x = k_y = k,$$ Support is symmetric

$$x = Xe^{i\omega t},$$

$$y = Ye^{i\omega t},$$

Simple harmonic motion

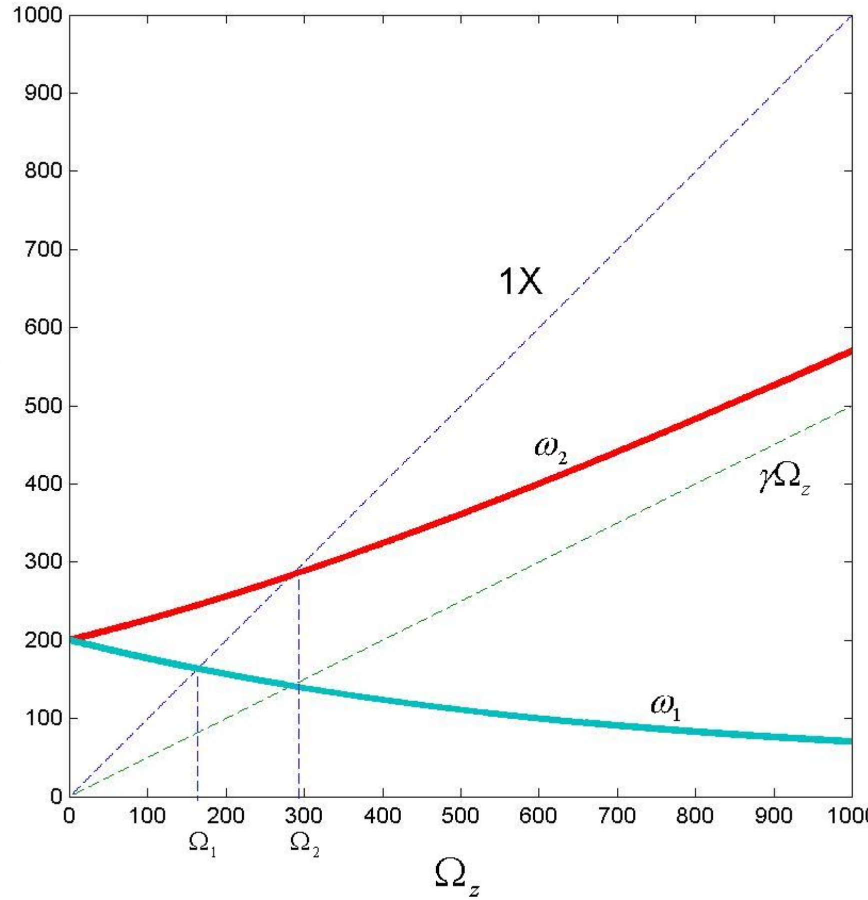
$$\begin{bmatrix} \left(k - \omega^2 I_0 / L^2\right) & \left(i\omega J\Omega_z / L^2\right) \\ \left(-i\omega J\Omega_z / L^2\right) & \left(k - \omega^2 I_0 / L^2\right) \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

$$\omega^4 - \left(2\frac{kL^2}{I_0} + \left(\frac{J\Omega_z}{I_0}\right)^2\right)\omega^2 + \left(\frac{kL^2}{I_0}\right)^2 = 0.$$

Natural Frequencies

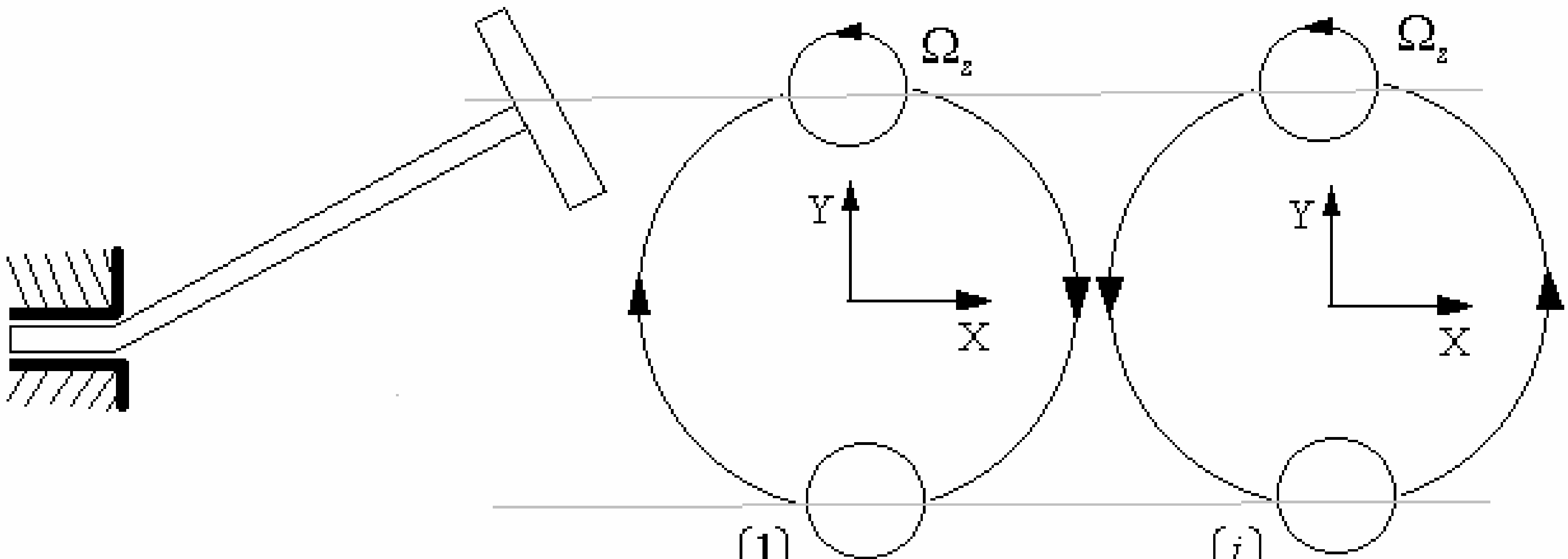
$$\omega^4 - \left(2\frac{kL^2}{I_0} + \left(\frac{J\Omega_z}{I_0} \right)^2 \right)\omega^2 + \left(\frac{kL^2}{I_0} \right)^2 = 0.$$

$$\Rightarrow \begin{cases} \omega_{1,2}^2 = \omega_0^2 + \dfrac{\gamma^2\Omega^2}{2} \pm \gamma\ \Omega\sqrt{\omega_0^2 + \dfrac{\gamma^2\Omega^2}{4}} \\ \omega_0^2 = \dfrac{kL^2}{I_0}, \qquad \gamma = \dfrac{J}{I_0} \end{cases}$$



Mode Shapes

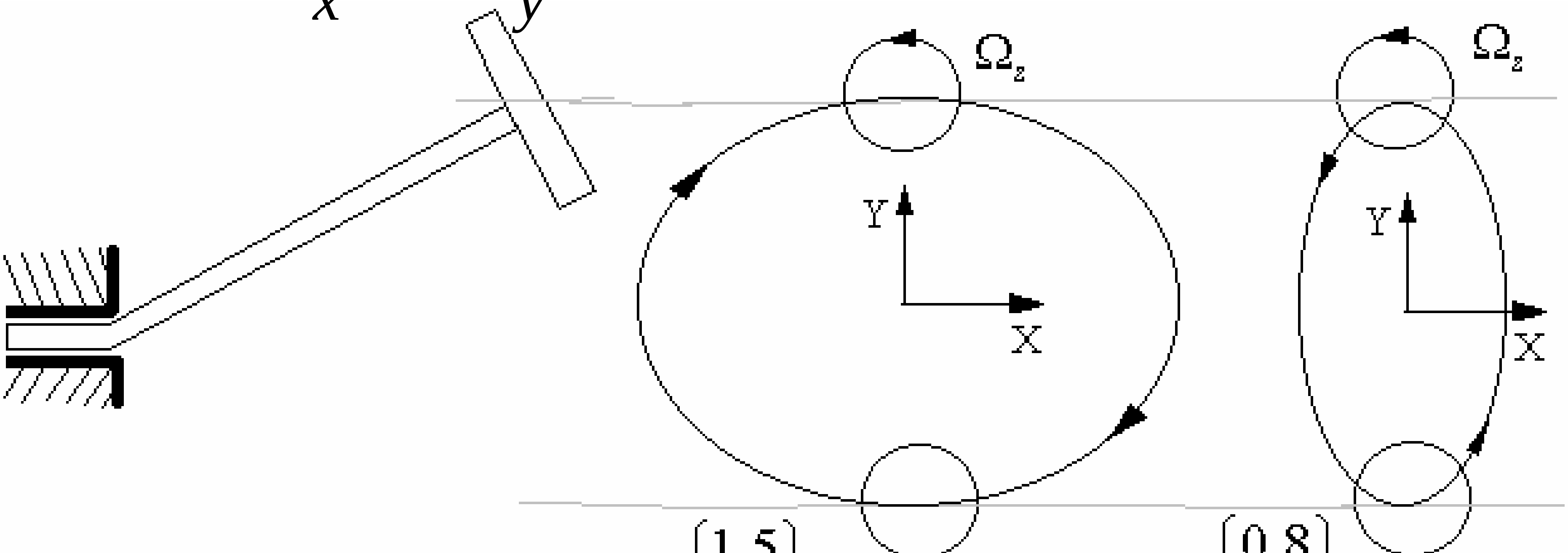
$$\begin{bmatrix} \left(k-\omega_1^2 I_0/L^2\right) & \left(i\omega_1 J\Omega_z/L^2\right) \\ \left(-i\omega_1 J\Omega_z/L^2\right) & \left(k-\omega_1^2 I_0/L^2\right) \end{bmatrix} \begin{Bmatrix} 1 \\ i \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{bmatrix} \left(k-\omega_2^2 I_0/L^2\right) & \left(i\omega_2 J\Omega_z/L^2\right) \\ \left(-i\omega_2 J\Omega_z/L^2\right) & \left(k-\omega_2^2 I_0/L^2\right) \end{bmatrix} \begin{Bmatrix} i \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$



$$\{\phi_1\} = \begin{Bmatrix} 1 \\ i \end{Bmatrix} \qquad \{\phi_2\} = \begin{Bmatrix} i \\ 1 \end{Bmatrix}$$

Non-symmetric Stator

$$k_x \neq k_y$$



$$\{\phi_1\} = \begin{Bmatrix} 1.5 \\ i \end{Bmatrix} \qquad \{\phi_2\} = \begin{Bmatrix} 0.8 \\ -i \end{Bmatrix}$$

FRF of the Rotating Structure

External Damping

$$\begin{bmatrix} I_0/L^2 & 0 \\ 0 & I_0/L^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} c & J\Omega_z/L^2 \\ -J\Omega_z/L^2 & c \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} f_x \\ f_y \end{Bmatrix},$$

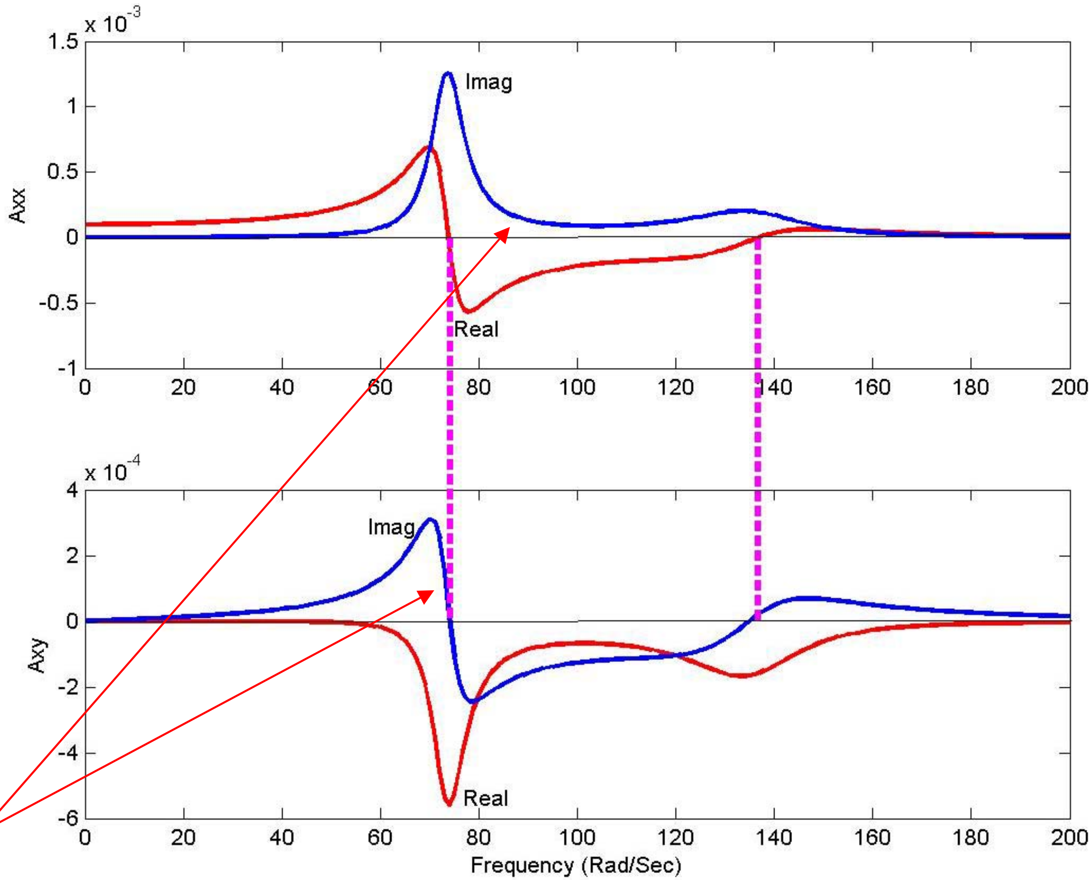
$$[\alpha(\omega)] = \begin{bmatrix} \left(k - \omega^2 I_0/L^2 + ic\omega\right) & \left(i\omega J\Omega_z/L^2\right) \\ -\left(i\omega J\Omega_z/L^2\right) & \left(k - \omega^2 I_0/L^2 + ic\omega\right) \end{bmatrix}^{-1}$$

$$\Rightarrow \begin{cases} \alpha_{xx}(\omega) = \alpha_{yy}(\omega) \\ \alpha_{xy}(\omega) = -\alpha_{yx}(\omega) \end{cases}$$

Loss of Reciprocity

Coupling Effect

FRF of the Rotating Structure with External Damping



Complex Mode Shapes due to significant imaginary part

Out-of-balance excitation

- Response analysis for the particular case of excitation provided by out-of-balance forces is investigated:
  - When the force results from an out-of-balance mass on the rotor, it is of a synchronous nature
  - When the force results from an out-of-balance mass on a co/counter rotating shaft, it is of a non-synchronous nature

Synchronous OOB Excitation

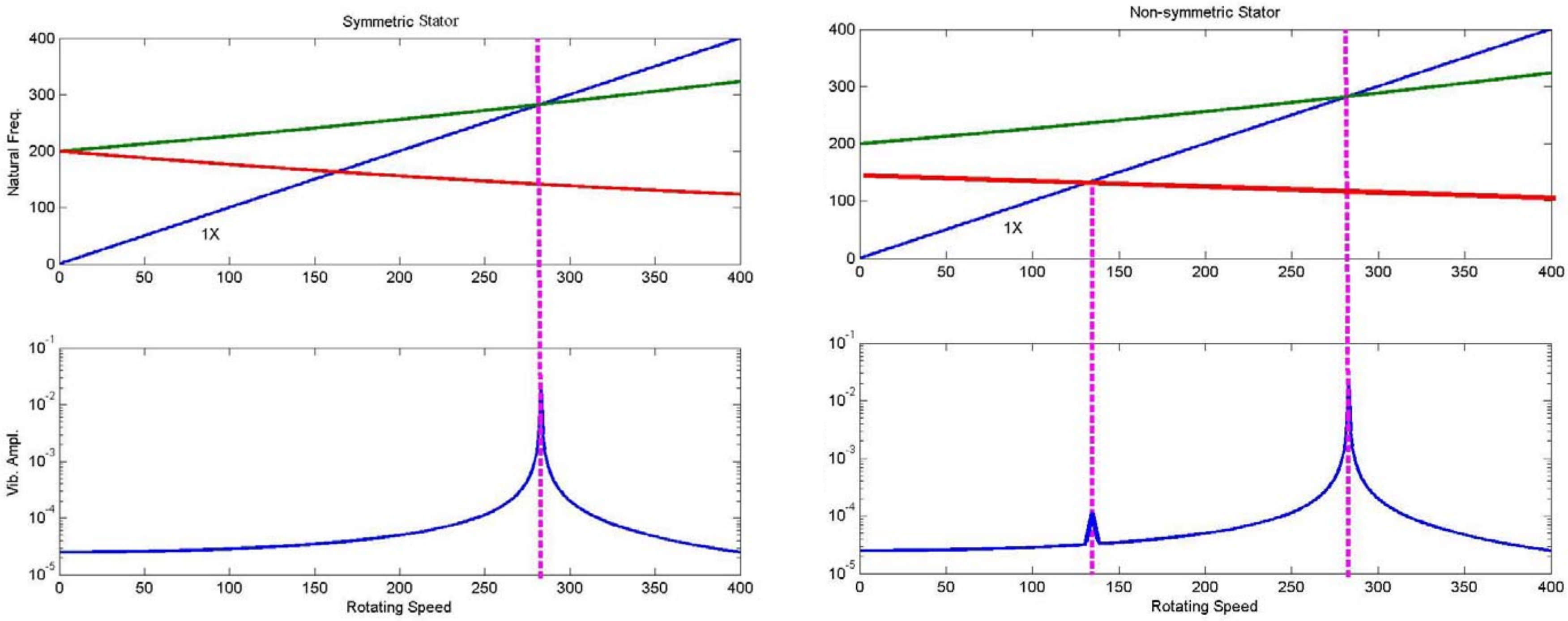
$$\{F\} = mr\Omega^2 \begin{Bmatrix} \cos(\Omega t) \\ \sin(\Omega t) \end{Bmatrix} = F_{OOB} \begin{Bmatrix} 1 \\ -i \end{Bmatrix} e^{i\Omega t}$$

$$Symmetric - Stator:$$

$$\Rightarrow \begin{Bmatrix} X \\ Y \end{Bmatrix} e^{i\Omega t} = F_{OOB} \begin{Bmatrix} A \\ -iA \end{Bmatrix} e^{i\Omega t}$$

$$A = \frac{L^2}{I_0\left(\omega_0^2 - \Omega^2(1-\gamma)\right)}$$

Synchronous OOB Excitation



Non-Synchronous OOB Excitation

- Force is generated by another rotor at different speed

$$Excitation \Rightarrow \beta\Omega$$

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} e^{i\beta\Omega t} = F_{OOB} \begin{Bmatrix} A \\ -iA \end{Bmatrix} e^{i\beta\Omega t}$$

$$A = \frac{L^2}{I_0\left(\omega_0^2 - \beta\Omega^2(\beta - \gamma)\right)}$$

- The essential results are the same as for synchronous case.

Modal Testing

(Lecture 6)

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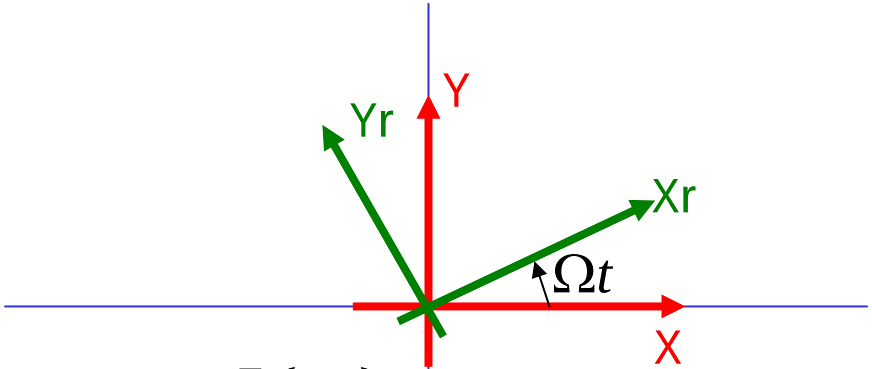
Theoretical Basis

- Analysis using rotating frame
- Damping in rotating and stationary frames
- Dynamic analysis of general rotor-stator systems
  - Linear Time Invariant Rotor-Stator Systems
  - LTI Rotor-Stator Viscous Damp System
  - LTI Systems Eigen-Properties

Analysis using rotating frame



$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos(\Omega t) & -\sin(\Omega t) \\ \sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{Bmatrix} x_r \\ y_r \end{Bmatrix}$$

$$\begin{Bmatrix} x_r \\ y_r \end{Bmatrix} = \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

Analysis using rotating frame

Transformation Matrices

$$\begin{Bmatrix} x_r \\ y_r \end{Bmatrix} = \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = [T_1] \begin{Bmatrix} x \\ y \end{Bmatrix},$$

$$\begin{Bmatrix} \dot{x}_r \\ \dot{y}_r \end{Bmatrix} = \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \Omega \begin{bmatrix} -\sin(\Omega t) & \cos(\Omega t) \\ -\cos(\Omega t) & -\sin(\Omega t) \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = [T_1] \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \Omega [T_2] \begin{Bmatrix} x \\ y \end{Bmatrix},$$

$$\begin{Bmatrix} \ddot{x}_r \\ \ddot{y}_r \end{Bmatrix} = \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + 2\Omega \begin{bmatrix} -\sin(\Omega t) & \cos(\Omega t) \\ -\cos(\Omega t) & -\sin(\Omega t) \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} - \Omega^2 \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$= [T_1] \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + 2\Omega [T_2] \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \Omega^2 [T_1] \begin{Bmatrix} x \\ x \end{Bmatrix}$$

Analysis using rotating frame

Equation of Motion in Stationary Coordinates

$$\begin{bmatrix} I_0/L^2 & 0 \\ 0 & I_0/L^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} 0 & J\Omega_z/L^2 \\ -J\Omega_z/L^2 & 0 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

$$\omega_1 = \sqrt{\omega_0^2 + (\gamma\Omega_z/2)^2} - \gamma\Omega_z/2 \qquad \omega_2 = \sqrt{\omega_0^2 + (\gamma\Omega_z/2)^2} + \gamma\Omega_z/2$$

Equation of Motion in Rotating Coordinates

$$\begin{bmatrix} I_0/L^2 & 0 \\ 0 & I_0/L^2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_r \\ \ddot{y}_r \end{Bmatrix} + \begin{bmatrix} 0 & -2\Omega_z I_0/L^2 + J\Omega_z/L^2 \\ 2\Omega_z I_0/L^2 - J\Omega_z/L^2 & 0 \end{bmatrix}\begin{Bmatrix} \dot{x}_r \\ \dot{y}_r \end{Bmatrix}$$

$$+\begin{bmatrix} -\Omega_z^2 I_0/L^2 + J\Omega_z^2/L^2 + k_x c^2 + k_y s^2 & cs(k_y - k_x) \\ cs(k_y - k_x) & -\Omega_z^2 I_0/L^2 + J\Omega_z^2/L^2 + k_x c^2 + k_y s^2 \end{bmatrix}\begin{Bmatrix} x_r \\ y_r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$\omega_1 = \sqrt{\omega_0^2 + (\gamma\Omega_z/2)^2} - \gamma\Omega_z/2 + \Omega_z \qquad \omega_2 = \sqrt{\omega_0^2 + (\gamma\Omega_z/2)^2} + \gamma\Omega_z/2 - \Omega_z$$

Note: Eigenvectors remain unchanged

Analysis using rotating frame

$$\begin{Bmatrix} F_{xr} \\ F_{yr} \end{Bmatrix} = \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}.$$

For Example:

Response harmonies not present in the excitation

$$\begin{Bmatrix} F_{xr} \\ F_{yr} \end{Bmatrix} = \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \cos(\omega t)$$

$$= \frac{F_0}{2} \begin{Bmatrix} \cos(\omega - \Omega)t + \cos(\omega + \Omega)t \\ \sin(\omega - \Omega)t + \sin(\omega + \Omega)t \end{Bmatrix}$$

Internal Damping in rotating and stationary frames

Equation of Motion in Rotating Coordinates

$$\begin{bmatrix} I_0/L^2 & 0 \\ 0 & I_0/L^2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_r \\ \ddot{y}_r \end{Bmatrix} + \begin{bmatrix} c_I & -2\Omega_z I_0/L^2 + J\Omega_z/L^2 \\ 2\Omega_z I_0/L^2 - J\Omega_z/L^2 & c_I \end{bmatrix}\begin{Bmatrix} \dot{x}_r \\ \dot{y}_r \end{Bmatrix}$$
$$+\begin{bmatrix} -\Omega_z^2 I_0/L^2 + J\Omega_z^2/L^2 + k & 0 \\ 0 & -\Omega_z^2 I_0/L^2 + J\Omega_z^2/L^2 + k \end{bmatrix}\begin{Bmatrix} x_r \\ y_r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Equation of Motion in Stationary Coordinates

$$\begin{bmatrix} I_0/L^2 & 0 \\ 0 & I_0/L^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} c_I & J\Omega_z/L^2 \\ -J\Omega_z/L^2 & c_I \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} k_x & \Omega_z c_I \\ -\Omega_z c_I & k_y \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

Internal/External Damping in 2DOF System

$$\begin{bmatrix} I_0/L^2 & 0 \\ 0 & I_0/L^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} c_E + c_I & J\Omega_z/L^2 \\ -J\Omega_z/L^2 & c_E + c_I \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix}$$

$$+\begin{bmatrix} k_x & \Omega_z c_I \\ -\Omega_z c_I & k_y \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

At super critical speeds the real parts of eigen-values may become positive, i.e. unstable system

Dynamic Analysis of General Rotor-Stator Systems

- The rotating machines and their modal testing is much more complex
  - Non-symmetric bearing support
  - Fixed/Rotating observation frame
  - Non-axisymmetric rotors
  - Internal/External damping
- These lead to:
  - Time-varying modal properties
  - Response harmonies not present in the excitation
  - Instabilites (negative modal damping)

Dynamic Analysis of General Rotor-Stator Systems

- Equation of motion of rotating systems are prone:
  - to lose the symmetry
  - to generate complex eigen-values/vectors from velocity/displacement related non-symmetry
  - to include time varying coefficients as appose to conventional Linear Time Invariant (LTI) systems

Dynamic Analysis of General Rotor-Stator Systems

System Type	Stationary Coord.	Rotating Coord.
R-symm;S-symm	LTI	LTI
R-symm;S-nonsymm	LTI	L(t)
R-nonsymm;S-symm	L(t)	LTI
R-nonsymm;S-nonsymm	L(t)	L(t)

LTI: Linear Time Invariant
L(t): Linear Time Dependent

Linear Time Invariant Rotor-Stator Systems

$$[M]\{\ddot{x}\}+([C]+[G(\Omega)])\{\dot{x}\}+$$

$$([K]+i[D]+[E(\Omega)])\{x\}=\{f(t)\}$$

$$[M],[C],[K],[D]\Rightarrow Symm.$$

$$[G(\Omega)],[E(\Omega)]\Rightarrow Skew-symm.$$

- Solution of equations will follow different routs depending upon the specific features.

LTI Rotor-Stator Systems (Viscous Damping Only)

$$[A]\{\dot{u}\}+[B]\{u\}=\{0\},$$

$$[A]=\begin{bmatrix} C+G(\Omega) & M \\ -M & 0 \end{bmatrix}$$

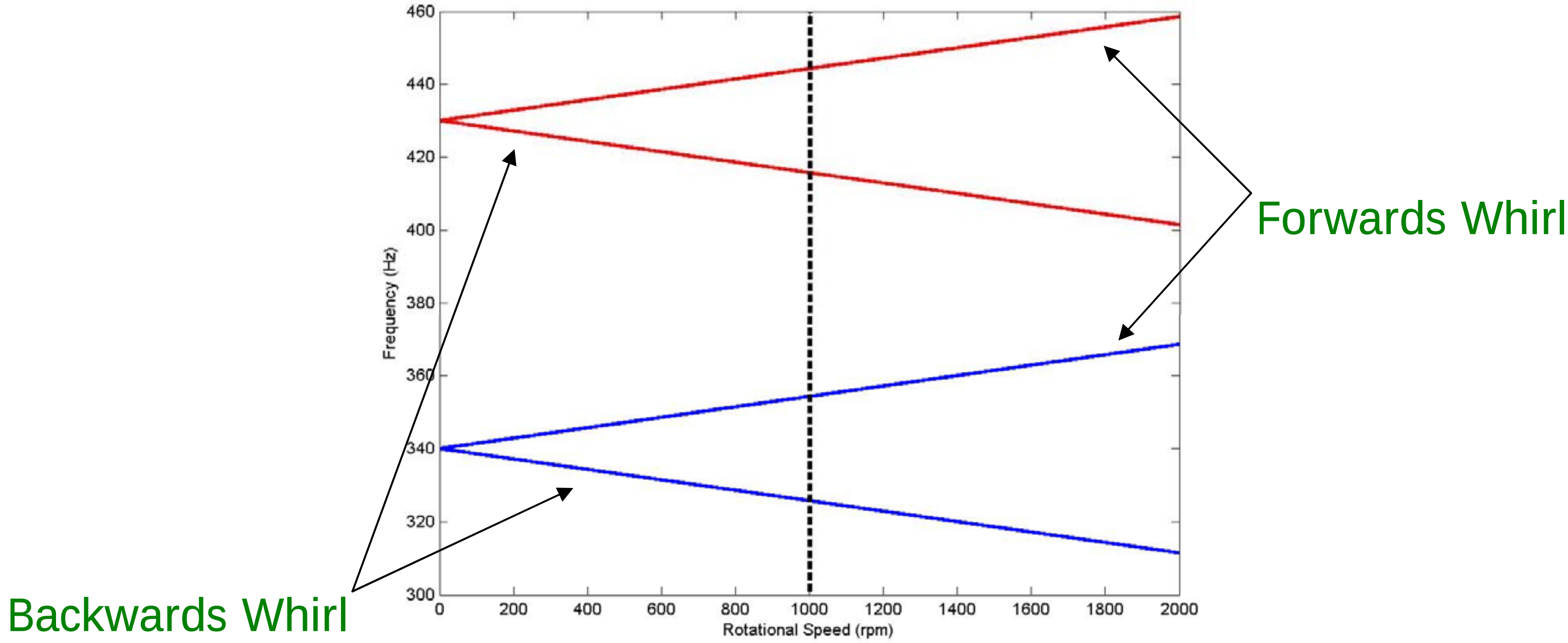
$$[B]=\begin{bmatrix} K+E(\Omega) & 0 \\ 0 & M \end{bmatrix}$$

$$\{u\}=\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$$

- The system matrices are non-symmetric
- Complex eigenvals
- Two eigenvect sets:
  - RH; mode shapes
  - LH; normal excitation shapes

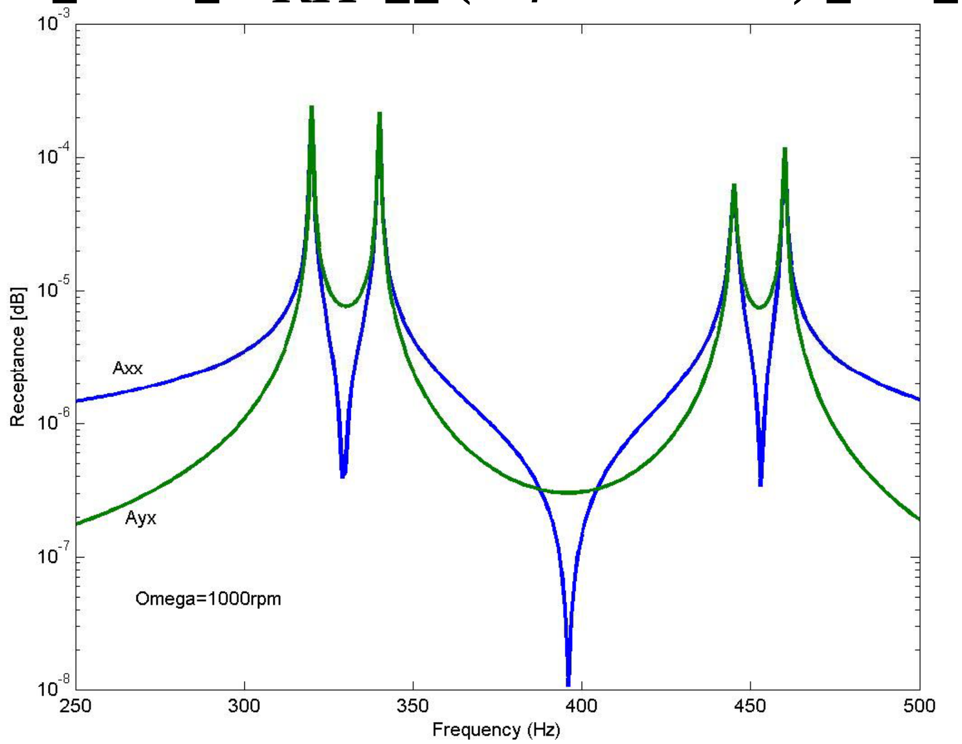
LTI Rotor-Stator Systems (Viscous Damping Only)

- Symmetric Rotor/ Non-symmetric Support



FRF of LTI Rotor-Stator Systems

$$[\alpha(\omega)] = [V_{RH}][(\lambda_r - i\omega)]^{-1}[V_{LH}]^H$$



LTI Systems Eigen-Properties

- Skew-symmetry in damping Matrix

$$[M]=\begin{bmatrix}1 & \\ & 1\end{bmatrix},[K]=\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix},$$

$$[C]=\Delta C\begin{bmatrix}0 & 0.5\\ -0.5 & 0\end{bmatrix}+(1-\Delta C)\begin{bmatrix}1 & -0.5\\ -0.5 & 1\end{bmatrix}$$

LTI Systems Eigen-Properties

ΔC	λ_2	X_1	X_2
0.0	-0.75+1.85i	1	-1.00
0.1	-0.68+1.88i	1	-1.05+0.08i
0.3	-0.52+1.94i	1	-1.08+0.28i
0.5	-0.37+1.99i	1	-1.03+0.49i
0.7	-0.23+2.04i	1	-0.90+0.63i
0.9	-0.07+2.08i	1	-0.76+0.71i
1.0	2.11i	1	-0.69+0.73i

LTI Systems Eigen-Properties

- Skew-symmetry in stiffness Matrix

$$[M] = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, [C] = 0,$$

$$[K] = \Delta K \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + (1 - \Delta K) \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

LTI Systems Eigen-Properties

ΔK	λ_2	X_1	X_2
0.0	2.00i	1	-1.00
0.1	1.90i	1	-1.12
0.3	1.65i	1	-1.58
0.5	1.23i	1	Infinity
0.7	0.32+1.00i	1	1.58i
0.9	0.57+0.79i	1	1.12i
1.0	0.70+0.70i	1	i

Modal Testing

(Lecture 7)

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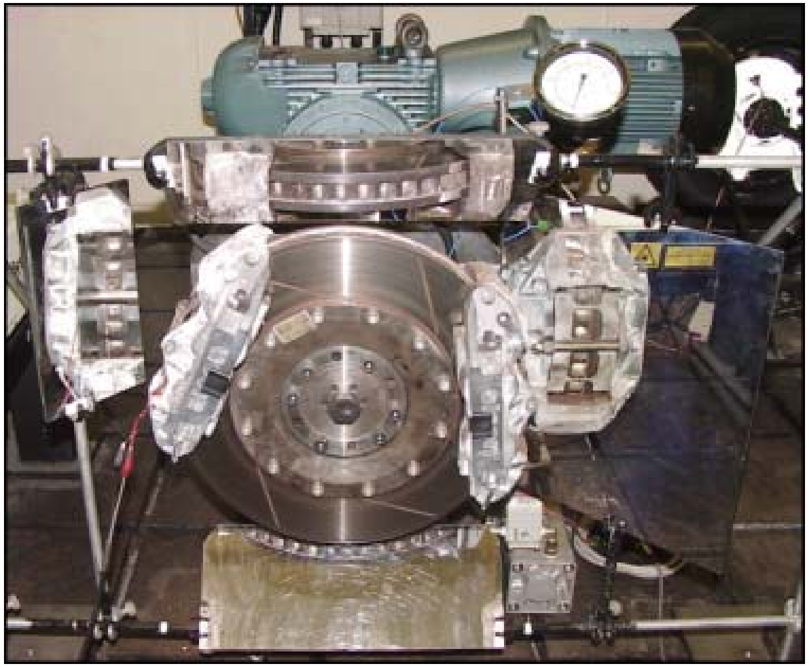
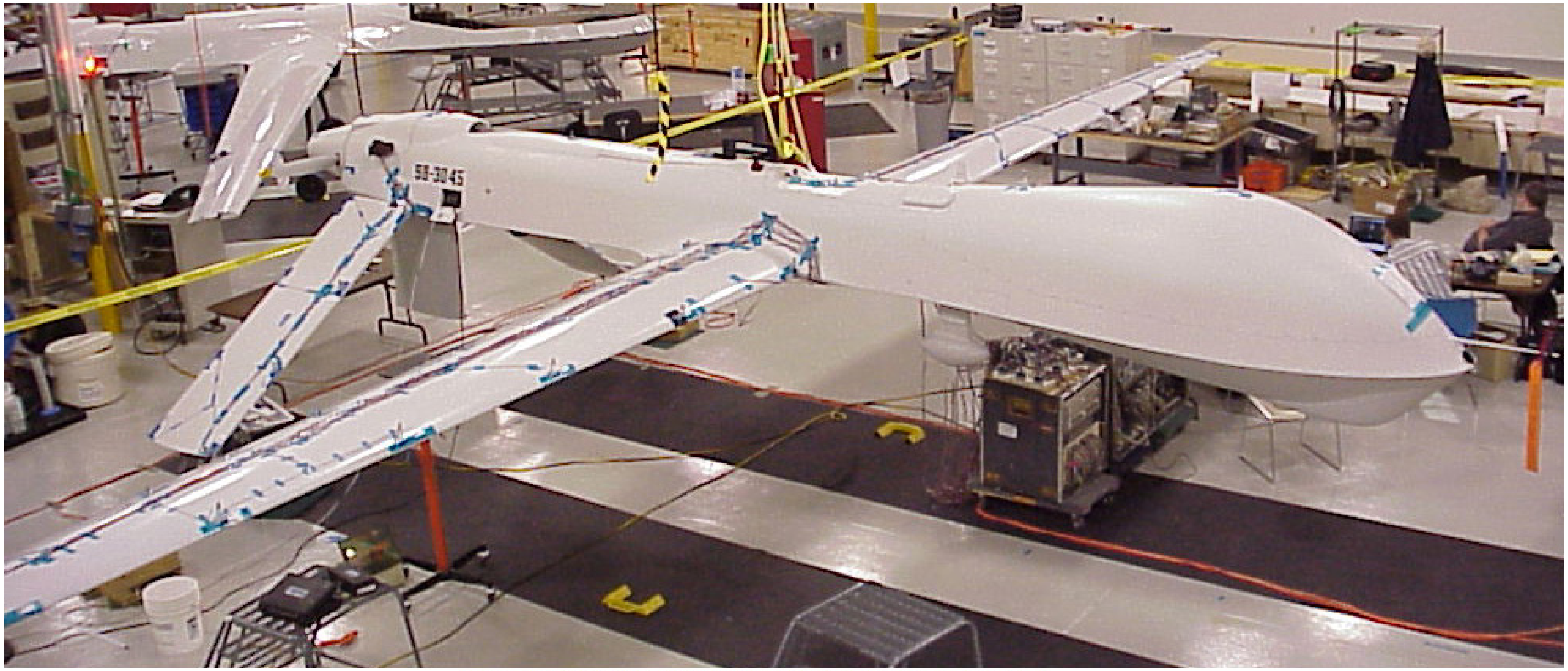
Complex Measured Modes

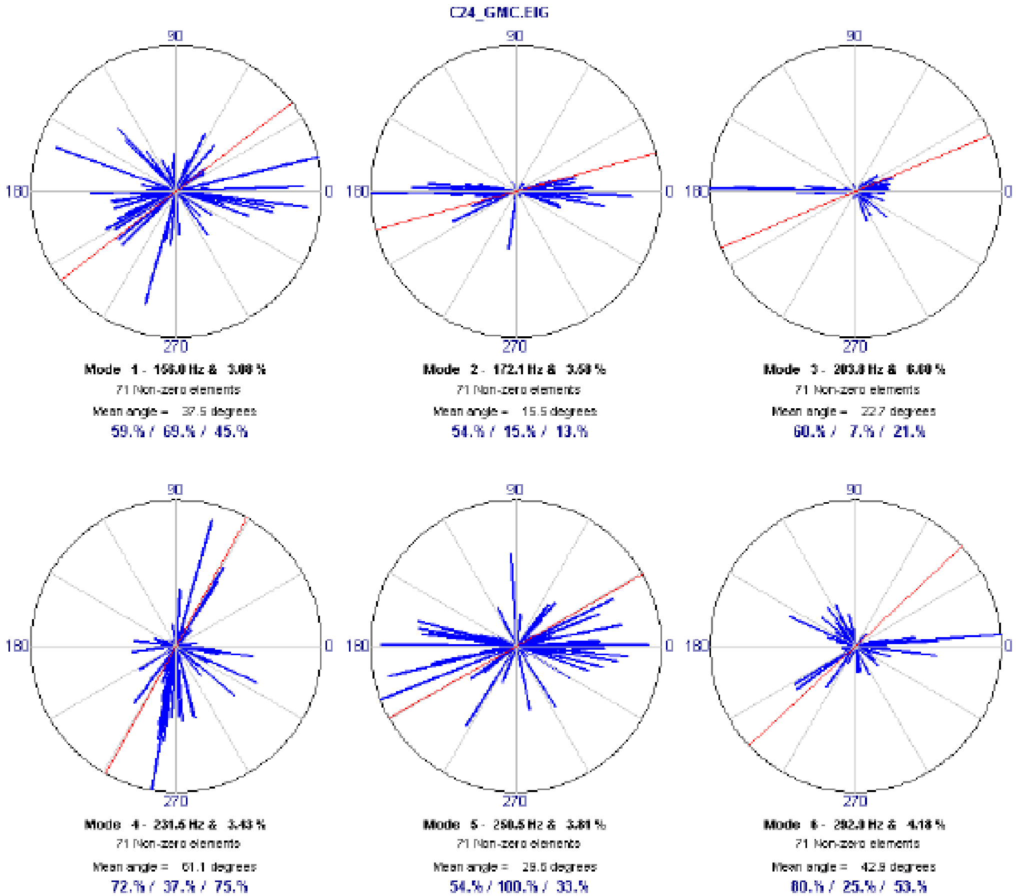


Complex Measured Modes

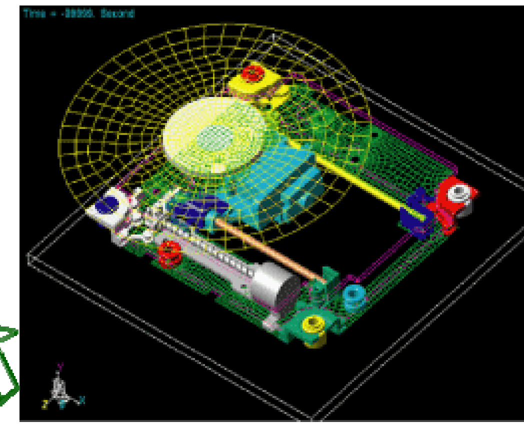


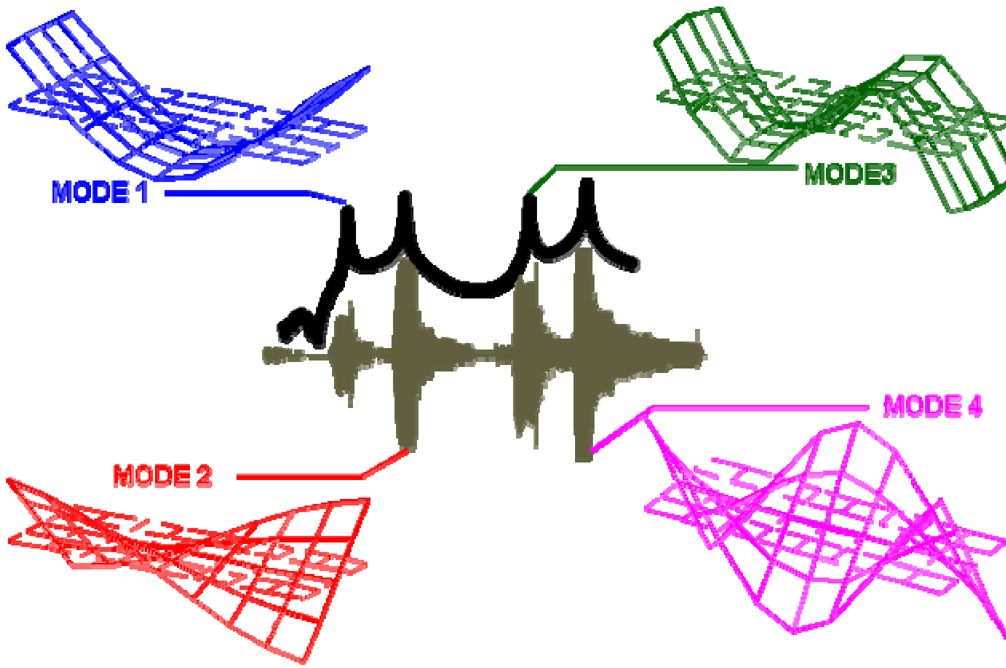
Predator Aircraft Ground Vibration Test
4 Shakers used at 8 Locations

Display of Mode Complexity



Analytical Real Modes





Extracting real modes from complex measured modes

- H Ahmadian, GML Gladwell - Proceedings of the 13th International Modal Analysis (1995):
  - The optimum real mode is the one with maximum correlation with the complex measured one:

$$max \quad \frac{|\,\phi_r^T \phi_c\,|}{\|\phi_r\|^2 \; \|\phi_c\|^2}.$$

Extracting real modes

- Normalizing the complex measured mode shape:

$$\|\phi_c\| = 1.$$

- The problem is rewritten as:

$$max \ \ (\phi_r^T \phi_c \phi_c^* \phi_r), \quad subject \ to \ \ \|\phi_r\| = 1.$$

Extracting real modes

Write $\phi_c = \phi_R + i\phi_I$, then

$$\phi_c \phi_c^* = U + iV,$$

where

Symmetric

Skew-symmetric

$$U = \phi_R \phi_R^T + \phi_I \phi_I^T, \qquad V = \phi_I \phi_R^T - \phi_R \phi_I^T.$$

Rank 2 matrices

Extracting real modes

- Since *V* is skew symmetric,

$$\phi_r^T V \phi_r = 0$$

- Therefore the problem is equivalent to:

$$max\ (\phi_r^T U \phi_r), \quad Subject\ to\ \|\phi_r\| = 1.$$

Extracting real modes

$$max\ (\phi_r^T U \phi_r), \quad Subject\ to\ \|\phi_r\| = 1.$$

But U is an $n \times n$ positive semi-definite matrix with rank 2. Therefore it has $(n-2)$ zero eigenvalues and 2 positive ones λ_1, and λ_2. The ϕ_r which maximizes (2) is the eigenvector corresponding to the larger of the two positive eigenvalues.

Extracting real modes

We now show that the real vector ϕ_r obtained as the eigenvector of U is precisely the same as the real part of the complex mode rotated so that its real part is maximized. To find this latter mode we must choose θ so that:

$$\max_{\theta} \| \, Real(\phi_c e^{i\theta}) \|^2 .$$

Extracting real modes

$$\| Real(\phi_c e^{i\theta})\|^2 = \|\phi_R \cos\theta + \phi_I \sin\theta\|^2,$$

$$= \phi_R^T\phi_R \cos^2\theta + \phi_I^T\phi_I \sin^2\theta + 2\phi_R^T\phi_I \sin\theta\cos\theta,$$

$$= \frac{\phi_R^T\phi_R + \phi_I^T\phi_I}{2} +$$

$$\{\frac{\phi_R^T\phi_R - \phi_I^T\phi_I}{2}\cos 2\theta + \phi_R^T\phi_I \sin 2\theta\},$$

so that the function is maximized or minimized when

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{\phi_R^T\phi_R - \phi_I^T\phi_I}{2\phi_R^T\phi_I}$$

Extracting real modes

To verify that the real part of the rotated mode, $\boldsymbol{\phi}_R \cos\theta + \boldsymbol{\phi}_I \sin\theta$, is an eigenvector of $\boldsymbol{U}$, i.e.

$$(\boldsymbol{\phi}_R\boldsymbol{\phi}_R^T + \boldsymbol{\phi}_I\boldsymbol{\phi}_I^T)(\boldsymbol{\phi}_R \cos\theta + \boldsymbol{\phi}_I \sin\theta) =$$

$$\lambda(\boldsymbol{\phi}_R \cos\theta + \boldsymbol{\phi}_I \sin\theta),$$

we note that this is true provided that:

$$\begin{aligned}(\boldsymbol{\phi}_R^T\boldsymbol{\phi}_R \cos\theta + \boldsymbol{\phi}_R^T\boldsymbol{\phi}_I \sin\theta) &= \lambda\cos\theta,\\ (\boldsymbol{\phi}_I^T\boldsymbol{\phi}_I \cos\theta + \boldsymbol{\phi}_I^T\boldsymbol{\phi}_R \sin\theta) &= \lambda\sin\theta.\end{aligned}$$

Extracting real modes

$$2(\phi_R^T\phi_I)\cos 2\theta = (\phi_R^T\phi_R - \phi_I^T\phi_I)\sin 2\theta,$$

$$(\phi_R^T\phi_I)(\cos^2\theta - \sin^2\theta) =$$

$$(\phi_R^T\phi_R - \phi_I^T\phi_I)\sin\theta\cos\theta,$$

$$(\phi_R^T\phi_R\cos\theta + \phi_R^T\phi_I\sin\theta)\sin\theta =$$

$$(\phi_R^T\phi_I\cos\theta + \phi_I^T\phi_I\sin\theta)\cos\theta.$$

This last equation implies that there is a constant λ satisfying equations (6), (7).

Follow-ups:

- E. Foltete, J. Piranda, “Transforming Complex Eigenmodes into Real Ones Based on an Appropriation Technique”, Journal of Vibration and Acoustics, JANUARY 2001, Vol. 123
- S.D. GARVEY, J.E.T. PENNY, “THE RELATIONSHIP BETWEEN THE REAL AND IMAGINARY PARTS OF COMPLEX MODES”, *Journal of Sound and Vibration* 1998,212(1),75-83

Modal Testing

(Lecture 8)

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Theoretical Basis

- Non-sinusoidal Vibration and FRF Properties:
  - Periodic Vibration
  - Transient Vibration
  - Random Vibration
    - Violation of Dirichlet's conditions
    - Autocorrelation and PSD functions
    - H1 and H2
- Incomplete Response Models

Non-sinusoidal Vibration and FRF Properties

- With the FRF data, response of a MDOF system to a set of harmonic loads:

$$\{X\}e^{i\omega t} = [\alpha(\omega)]\{F\}e^{i\omega t}$$

The same frequency

Different amplitudes and phases

- We shall now turn our attention to a range of other excitation/response situvations.

Periodic Vibration

- Excitation is not simply sinusoidal but retain periodicity.
- The easiest way of computing the response is by means of Fourier Series,

$$f_k(t) = \sum_{n=1}^{\infty} F_{nk} e^{i\omega_n t} \qquad \omega_n = \frac{2\pi}{T}$$

$$x_j(t) = \sum_{n=1}^{\infty} \alpha_{jk}(\omega_n) F_{nk} e^{i\omega_n t}$$

Periodic Vibration

- To derive FRF from periodic vibration signals:
  - Determine the Fourier Series components of the input force and the relevant response
  - Both series contain components at the same set of discrete frequencies
  - The FRF can be defined at the same set of frequency points by computing the ratio of response to input components.

Transient Vibration

- Analysis via Fourier Transform

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

$$X(\omega) = H(\omega) F(\omega)$$

$$x(t) = \int_{-\infty}^{+\infty} H(\omega) F(\omega) e^{i\omega t} d\omega$$

Transient Vibration

- Response via time domain (superposition)

$$x(t) = \int_{-\infty}^{+\infty} h(t-\tau) f(\tau) d\tau$$

$$Let \rightarrow f(t) = \delta(0) \Rightarrow F(\omega) = \frac{1}{2\pi}$$

$$Then \rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega) e^{i\omega t} d\omega = h(t)$$

Transient Vibration

- To derive FRF from transient vibration signals:
  - Calculation of the Fourier Transforms of both excitation and response signals
  - Computing the ratio of both signals at the same frequency
- In practice it is common to compute a DFT of the signals.

Random Vibration

- Neither excitation nor response signal can be subject to a valid Fourier Transform:
  - Violation of Dirichlet Conditions
    - Finite number of isolated min and max
    - Finite number of points of finite discontinuity
- Here we assume the random signals to be ergodic

Random Vibration

$$f(t)$$ ← **Time Signal**

Autocorrelation Function

$$R_{ff}(\tau) = \int_{-\infty}^{+\infty} f(t) f(t+\tau) dt$$

Power Spectral Density

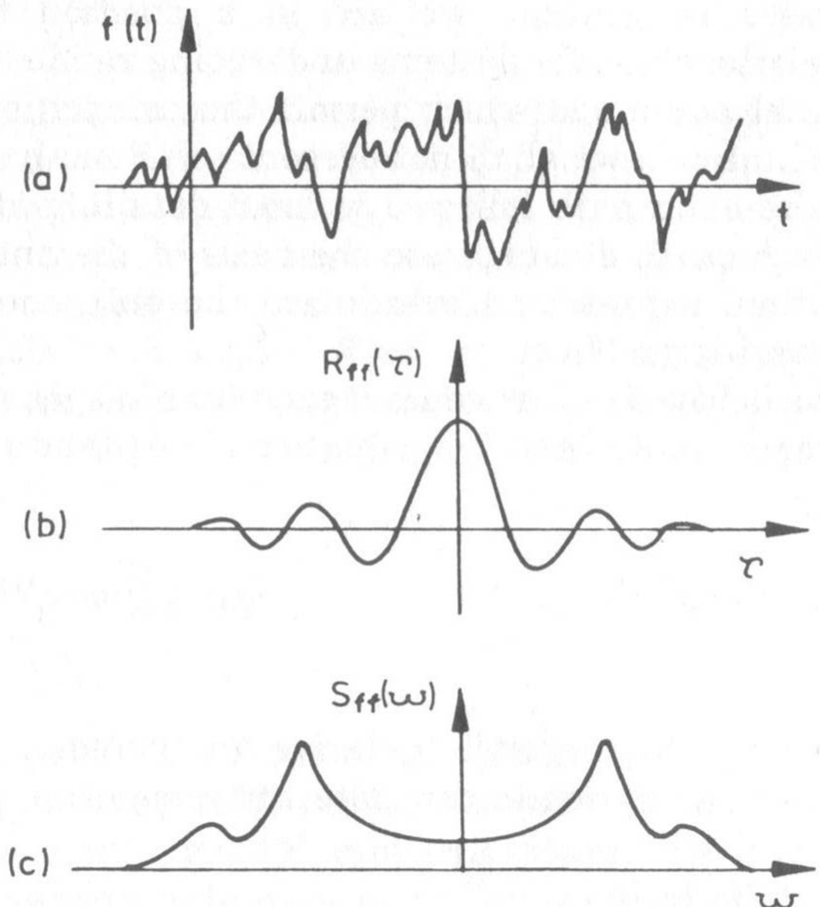
$$S_{ff}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{ff}(\tau) e^{-i\omega\tau} d\tau$$

Random Vibration

Time Signal

Autocorrelation

Power Spectral Density

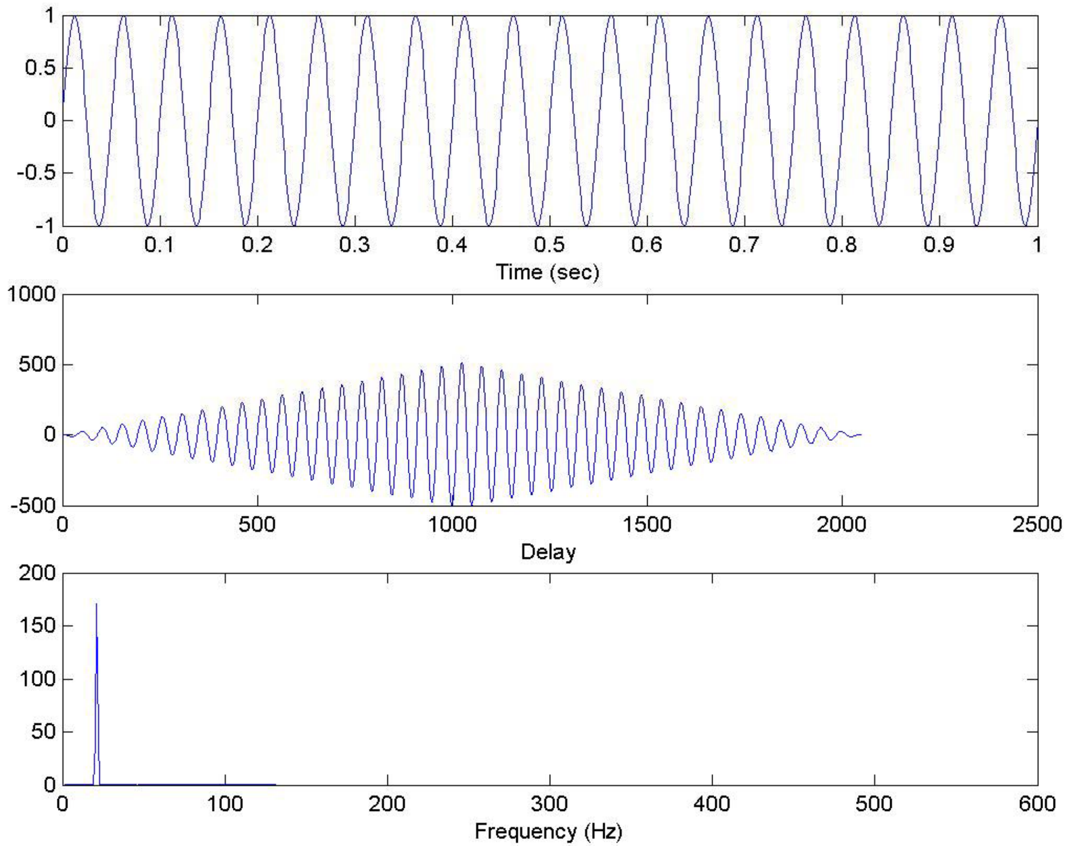


Random Vibration

Sinusoidal Signal

Autocorrelation

Power Spectral Density

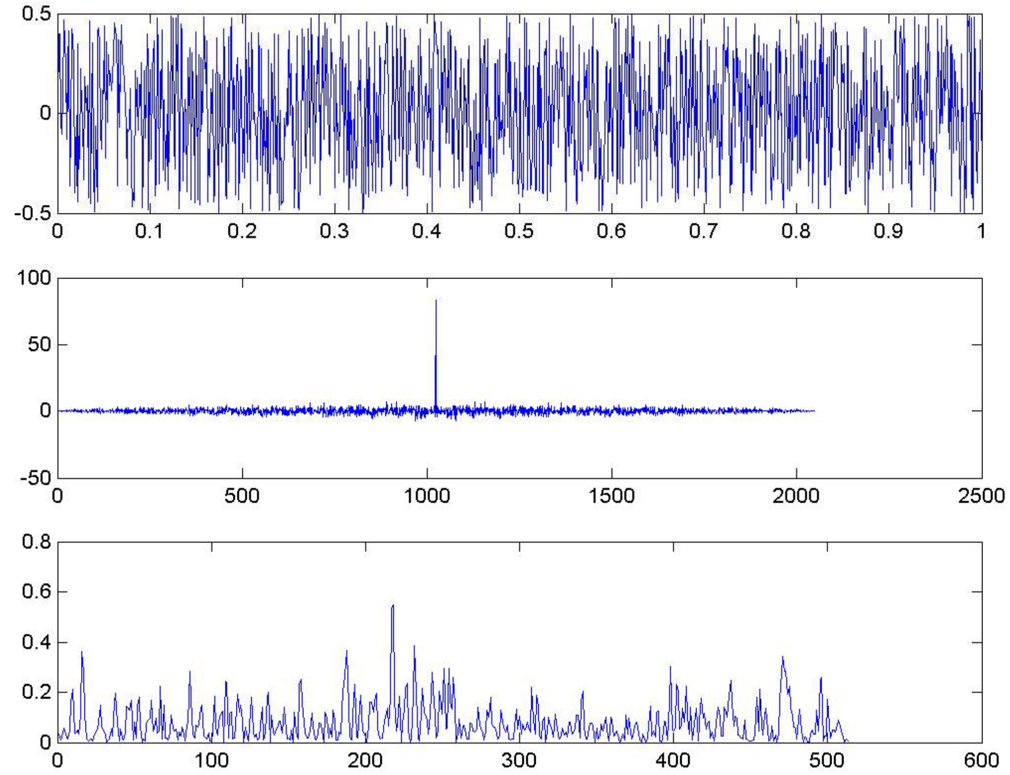


Random Vibration

Random Signal

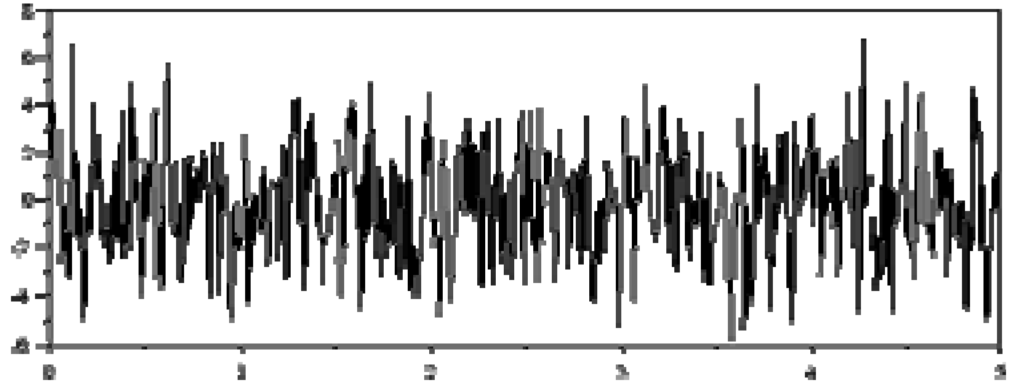
Autocorrelation

Power Spectral Density

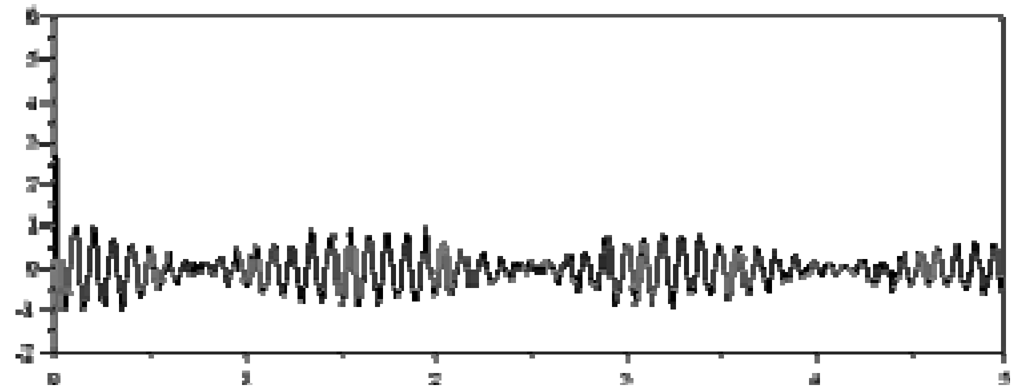


Random Vibration

Noisy Signal



Autocorrelation



Random Vibration

- The autocorrelation function is real and even:

$$R_{ff}(\tau) = \int_{-\infty}^{+\infty} f(t) f(t+\tau) dt$$

$$= \int_{-\infty}^{+\infty} f(u-\tau) f(u) du = R_{ff}(-\tau)$$

$$u = t + \tau$$

- The Auto/Power Spectral Density function is real and even.

Random Vibration

- Cross Correlation / Spectral Densities

$$R_{xf}(\tau) = \int_{-\infty}^{+\infty} x(t) f(t+\tau) dt \qquad S_{xf}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{xf}(\tau) e^{-i\omega\tau} d\tau$$

- Cross Correlation functions are real but not always even.
- Cross Spectral Densities are complex functions.

Random Vibration

Time Domain **Frequency Domain**

$$R_{ff}(\tau) = \int_{-\infty}^{+\infty} f(t) f(t+\tau) dt \Rightarrow S_{ff}(\omega) = F^{*}(\omega) F(\omega)$$

$$R_{xf}(\tau) = \int_{-\infty}^{+\infty} x(t) f(t+\tau) dt \Rightarrow S_{xf}(\omega) = X^{*}(\omega) F(\omega)$$

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} x(t) x(t+\tau) dt \Rightarrow S_{xx}(\omega) = X^{*}(\omega) X(\omega)$$

Random Vibration

- To derive FRF from random vibration signals:

$$H(\omega) = \frac{X(\omega)}{F(\omega)}$$

$$H_1(\omega) = \frac{X^*(\omega)X(\omega)}{X^*(\omega)F(\omega)} = \frac{S_{xx}(\omega)}{S_{xf}(\omega)}$$

$$H_2(\omega) = \frac{F^*(\omega)X(\omega)}{F^*(\omega)F(\omega)} = \frac{S_{fx}(\omega)}{S_{ff}(\omega)}$$

Complete/ Incomplete Models

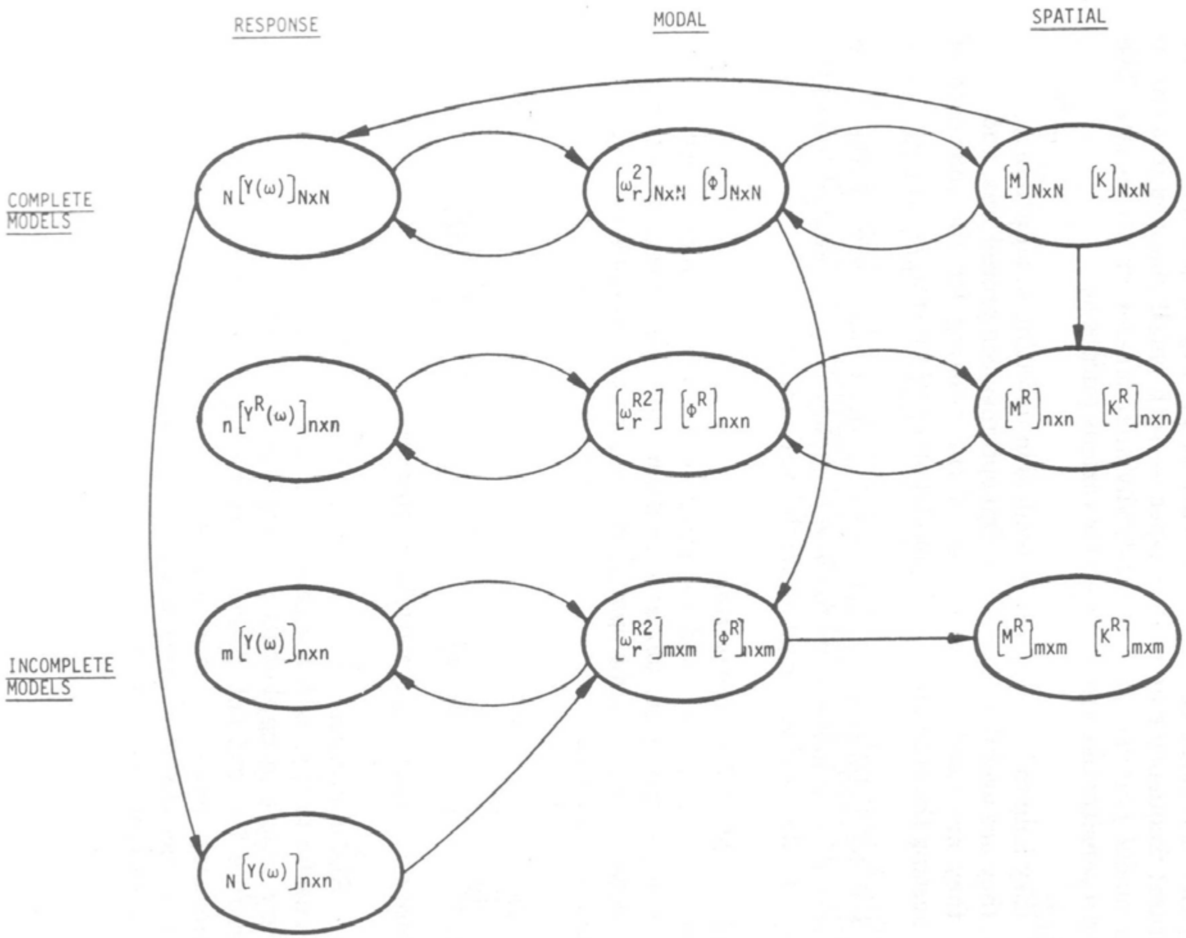
- It is not possible to measure the response at all DOF or all modes of structure (N by N)
- Different incomplete models:
  - Reduced size (from N to n) by deleting some DOFs
  - Number of modes are a reduced as well (from N to m, usually $m<n$)

Incomplete Response Models

$$\alpha_{jk}(\omega) = \sum_{r=1}^{m<N} \frac{{}_rA_{jk}}{\omega_r^2 - \omega^2 + i\eta_r\omega_r^2}$$

$$\Rightarrow \begin{cases} \left[\omega_r^2\right]_{m\times m} \\ \left[\Phi\right]_{n\times m} \end{cases}$$

Incomplete Response Models



Modal Testing

(Lecture 9)

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Theoretical Basis

- Sensitivity of Models
  - Modal Sensitivity
    - SDOF eigen sensitivity
    - MDOF system natural frequency sensitivity
    - MDOF system mode shape sensitivity
  - FRF Sensitivity
    - SDOF FRF sensitivity
    - MDOF FRF sensitivity

Sensitivity of Models

- The sensitivity analysis are required:
  - to help locate errors in models in updating
  - to guide design optimization procedures
  - they are used in the course of curve fitting
- A short summery on deducing sensitivities from experimental and analytical models is given.

Modal Sensitivities (SDOF)

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\frac{\partial \omega_0}{\partial m} = -\frac{1}{2}\sqrt{\frac{k}{m^3}} = -\frac{1}{2}\frac{\omega_0}{m},$$

$$\frac{\partial \omega_0}{\partial k} = \frac{1}{2\sqrt{mk}} = \frac{1}{2}\frac{\omega_0}{k}.$$

Modal Sensitivities (MDOF)

$$\left([K]-\omega_r^2[M]\right)\{\phi_r\}=\{0\},$$

$$\frac{\partial}{\partial p}\left([K]-\omega_r^2[M]\right)\{\phi_r\}=\{0\},$$

$$\left([K]-\omega_r^2[M]\right)\frac{\partial\{\phi_r\}}{\partial p}+\left(\frac{\partial[K]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}[M]-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\},$$

Eigenvalue Sensitivity (MDOF)

$$Multiply\ by \quad \{\phi_r\}^T$$

$$\{\phi_r\}^T\left([K]-\omega_r^2[M]\right)\frac{\partial\{\phi_r\}}{\partial p}+$$

$$\{\phi_r\}^T\left(\frac{\partial[K]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}[M]-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\},$$

$$results \Rightarrow \frac{\partial\omega_r^2}{\partial p}=\frac{\{\phi_r\}^T\left(\frac{\partial[K]}{\partial p}-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}}{\{\phi_r\}^T[M]\{\phi_r\}}$$

Eigenvector Sensitivity (MDOF)

$Starting \quad from:$

$$\left([K]-\omega_r^2[M]\right)\frac{\partial\{\phi_r\}}{\partial p}+\left(\frac{\partial[K]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}[M]-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\},$$

$and\ taking \quad \frac{\partial\{\phi_r\}}{\partial p}=\sum_{\substack{j=1\\ j\neq r}}^{N}\gamma_j\{\phi_j\}$

$$\Rightarrow\left([K]-\omega_r^2[M]\right)\sum_{\substack{j=1\\ j\neq r}}^{N}\gamma_{rj}\{\phi_j\}+\left(\frac{\partial[K]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}[M]-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\}$$

Eigenvector Sensitivity (MDOF)

$$\left([K]-\omega_r^2[M]\right)\sum_{\substack{j=1\\ j\neq r}}^{N}\gamma_{rj}\{\phi_j\}+\left(\frac{\partial[K]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}[M]-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\}$$

$$\Rightarrow\{\phi_s\}^T\left([K]-\omega_r^2[M]\right)\sum_{\substack{j=1\\ j\neq r}}^{N}\gamma_{rj}\{\phi_j\}+\{\phi_s\}^T\left(\frac{\partial[K]}{\partial p}-\frac{\partial\omega_r^2}{\partial p}[M]-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\}$$

$$\Rightarrow\left(\omega_s^2-\omega_r^2\right)\gamma_{rs}+\{\phi_s\}^T\left(\frac{\partial[K]}{\partial p}-\omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}=\{0\}$$

Eigenvector Sensitivity (MDOF)

$$\Rightarrow \left(\omega_s^2 - \omega_r^2\right)\gamma_{rs} + \{\phi_s\}^T \left(\frac{\partial [K]}{\partial p} - \omega_r^2 \frac{\partial [M]}{\partial p} \right) \{\phi_r\}$$

$$\Rightarrow \gamma_{rs} = \frac{\{\phi_s\}^T \left(\frac{\partial [K]}{\partial p} - \omega_r^2 \frac{\partial [M]}{\partial p} \right) \{\phi_r\}}{\left(\omega_r^2 - \omega_s^2\right)}$$

$$\Rightarrow \quad \frac{\partial \{\phi_r\}}{\partial p} = \sum_{\substack{s=1 \\ s \neq r}}^{N} \frac{\{\phi_s\}^T \left(\frac{\partial [K]}{\partial p} - \omega_r^2 \frac{\partial [M]}{\partial p} \right) \{\phi_r\}}{\left(\omega_r^2 - \omega_s^2\right)} \{\phi_s\}$$

Updating, Redesign, Reanalysis

$$\begin{Bmatrix} \Delta\omega_1^2 \\ \Delta\omega_2^2 \\ \vdots \\ \{\Delta\phi_1\} \\ \{\Delta\phi_2\} \\ \vdots \end{Bmatrix} = \begin{bmatrix} \frac{\partial\omega_1^2}{\partial p_1} & \frac{\partial\omega_1^2}{\partial p_2} & \frac{\partial\omega_1^2}{\partial p_3} & \cdots \\ \frac{\partial\omega_2^2}{\partial p_1} & \frac{\partial\omega_2^2}{\partial p_2} & \frac{\partial\omega_2^2}{\partial p_3} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \frac{\partial\{\Delta\phi_1\}}{\partial p_1} & \frac{\partial\{\Delta\phi_1\}}{\partial p_2} & \frac{\partial\{\Delta\phi_1\}}{\partial p_3} & \cdots \\ \frac{\partial\{\Delta\phi_2\}}{\partial p_1} & \frac{\partial\{\Delta\phi_2\}}{\partial p_2} & \frac{\partial\{\Delta\phi_2\}}{\partial p_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \Delta p_3 \\ \vdots \end{Bmatrix}$$

Updating, Redesign, Reanalysis

- The change in parameters must be very small for accurate analysis
- When the change in parameters is not small:
  - Higher order sensitivity analysis
  - Iterative linear sensitivity analysis

FRF Sensitivities (SDOF)

$$\alpha(\omega) = \frac{1}{k + i\omega c - \omega^2 m}$$

$$\frac{\partial \alpha(\omega)}{\partial k} = \frac{-1}{\left(k + i\omega c - \omega^2 m\right)^2}$$

$$\frac{\partial \alpha(\omega)}{\partial c} = \frac{-i\omega}{\left(k + i\omega c - \omega^2 m\right)^2}$$

$$\frac{\partial \alpha(\omega)}{\partial m} = \frac{\omega^2}{\left(k + i\omega c - \omega^2 m\right)^2}$$

FRF Sensitivities (MDOF)

$$[Z(\omega)] = [K] + i\omega[C] - \omega^2[M],$$

$$\Rightarrow ([A]+[B])^{-1} = [A]^{-1} - ([A]+[B])^{-1}[B][A]^{-1}$$

$$take\,[A] \Rightarrow \quad [Z(\omega)]_A, \qquad\qquad [A+B] \Rightarrow \quad [Z(\omega)]_x$$

$$then \Rightarrow [Z(\omega)]_x^{-1} = [Z(\omega)]_A^{-1} - [Z(\omega)]_x^{-1}([Z(\omega)]_x - [Z(\omega)]_A)^{-1}[Z(\omega)]_A^{-1}$$

$$[\alpha(\omega)]_x - [\alpha(\omega)]_A = -[\alpha(\omega)]_x[\Delta Z(\omega)][\alpha(\omega)]_A$$

FRF Sensitivities (MDOF)

$$\left[\alpha(\omega)\right]_x - \left[\alpha(\omega)\right]_A = -\left[\alpha(\omega)\right]_x \left[\Delta Z(\omega)\right]\left[\alpha(\omega)\right]_A,$$

$$\left\{\alpha_x(\omega) - \alpha_A(\omega)\right\}_j^T = \left\{\alpha_x(\omega)\right\}_j^T \left[\Delta Z(\omega)\right]\left[\alpha(\omega)\right]_A$$

FRF Sensitivities (MDOF)

Starting with the analytical receptance matrix $[\alpha(\omega)]_A$, denoted as $[\alpha_A]$

$$[\alpha_A] = [\alpha_A]. \tag{1}$$

Adding and subtracting the experimental receptance matrix $[\alpha_X]$ to the right hand side of (1) gives:

$$[\alpha_A] = [\alpha_X] + [\alpha_A] - [\alpha_X]. \tag{2}$$

Multiplying $[\alpha_A]$ of the right hand side by $[\mathbf{I}] = [\alpha_X]^{-1}[\alpha_X]$

$$[\alpha_A] = [\alpha_X] + [\alpha_A][\alpha_X]^{-1}[\alpha_X] - [\alpha_X] \tag{3}$$

and factorising by $[\alpha_X]$ yields:

$$[\alpha_A] = [\alpha_X] + ([\alpha_A][\alpha_X]^{-1} - [\mathbf{I}])[\alpha_X]. \tag{4}$$

Replacing $[\mathbf{I}]$ by $[\alpha_A][\alpha_A]^{-1}$

$$[\alpha_A] = [\alpha_X] + ([\alpha_A][\alpha_X]^{-1} - [\alpha_A][\alpha_A]^{-1})[\alpha_X] \tag{5}$$

and factorising by $[\alpha_A]$ gives:

$$[\alpha_A] = [\alpha_X] + [\alpha_A]([\alpha_X]^{-1} - [\alpha_A]^{-1})[\alpha_X]. \tag{6}$$

Or, in a more familiar form,

$$[\alpha_A] - [\alpha_X] = [\alpha_A][\Delta \mathbf{Z}][\alpha_X] \tag{7}$$

where

$$[\Delta \mathbf{Z}] = [\mathbf{Z}_X] - [\mathbf{Z}_A] = [\Delta \mathbf{K}] - \omega^2[\Delta \mathbf{M}]. \tag{8}$$

FRF Sensitivities (MDOF)

$$\frac{\partial[\alpha(\omega)]}{\partial p} = \frac{\partial\left([Z(\omega)]^{-1}\right)}{\partial p} = -[Z(\omega)]^{-1}\frac{\partial[Z(\omega)]}{\partial p}[Z(\omega)]^{-1}$$

$$\frac{\partial[\alpha(\omega)]}{\partial p} = -[\alpha(\omega)]\frac{\partial[Z(\omega)]}{\partial p}[\alpha(\omega)]$$

$$\frac{\partial[\alpha(\omega)]}{\partial p} = -[\alpha(\omega)]\left(\frac{\partial[K]}{\partial p} + i\omega\frac{\partial[C]}{\partial p} - \omega^2\frac{\partial[M]}{\partial p}\right)[\alpha(\omega)]$$